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OSCILLATORY PROPERTIES OF VOLTERRA INTEGRAL EQUATIONS

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1. Introduction

Consider the Volterra integral equation with advanced argument

$$x(t) = f(t) - \int_0^t a(t,s)g(s,x(\tau(s)))ds, \quad t \geq 0. \tag{1}$$

In (1), $f : [0,\infty) \to R$ is continuous, $g : [0,\infty) \times R \to R$ is continuous, $\tau(t)$ is continuous, nondecreasing and $\tau(t) \geq t$ on $[0,\infty)$ and $a : [0,\infty) \times [0,\infty) \to R$ is such that $a(t,s) = 0$ if $s > t$, $a(t,s) \geq 0$ for $0 \leq t < \infty$ and $0 \leq s \leq t$. Let $a(t,s)$ be continuous for $0 \leq t < \infty$ and $0 \leq s \leq t$. We consider only the solutions of (1) which exist and continuous on $[0,\infty)$, and are nontrivial in any neighbourhood of infinity. A solution $x(t)$ of (1) is said to be *oscillatory* if each of the sets $\{t \geq 0 | x(t) > 0\}$ and $\{t > 0 | x(t) < 0\}$ is unbounded; it is said to be *weakly oscillatory* if the set $\{t \geq 0 | x(t) = 0\}$ is unbounded; and it is said to be *nonoscillatory* if it is not weakly oscillatory. We note that this notion of *weakly oscillatory* is usually called *oscillatory* (Cf. [1]), but here we use the definition of oscillatory as same as in [2]. In this paper, we propose some criteria sufficient to imply all solutions of (1) are weakly oscillatory, which is not considered in [2].

2. Results

Theorem 1. *Let* $\limsup_{t\to\infty} f(t) = M$ *and* $\liminf_{t\to\infty} f(t) = N$, *where* $M > 0$ *and* $N < 0$ *are constants. Let* $xg(t,x) > 0$ *if* $x \neq 0$ *and the function* $h_\sigma(t) = \int_0^\sigma a(t,s)ds$ *satisfies* $\lim_{t\to\infty} h_\sigma(t) = 0$ *for every fixed* $\sigma > 0$. *Then all solutions of (1) are weakly oscillatory.*

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Proof. Let $x(t)$ be a not weakly oscillatory solution of (1) on $[0,\infty)$. Then there exists a $T > 0$ such that $x(t) > 0$ or < 0 for sufficiently large t. Suppose that $x(t) > 0$ for $t \geq T > 0$. From (1) and $\tau(t) \geq t$, we obtain $x(\tau(t)) > 0$ on $t \geq T$ and

$$\begin{aligned} 0 < x(t) &= f(t) - \int_0^t a(t,s)g(s,x(\tau(s)))ds \\ &= f(t) - \int_0^T a(t,s)g(s,x(\tau(s)))ds \\ &\quad - \int_T^t a(t,s)g(s,x(\tau(s)))ds \\ &\leq f(t) - \int_0^T a(t,s)g(s,x(\tau(s)))ds \qquad (2) \\ &\leq f(t) + L\int_0^T a(t,s)ds, \quad T \leq t < \infty \\ &\quad \text{where } L = \sup_{t\in[0,T]} |g(t,x(\tau(t)))|. \end{aligned}$$

This (2), $\liminf_{t\to\infty} f(t) = N < 0$ and $\lim_{t\to\infty} h_T(t) = \int_0^T a(t,s)ds = 0$ lead a contradiction to $x(t) > 0$. On the other hand, suppose $x(t) < 0$ for $t \geq T^* > 0$. From (1), we have

$$\begin{aligned} 0 > x(t) &= f(t) - \int_0^{T^*} a(t,s)g(s,x(\tau(s)))ds \\ &\quad - \int_{T^*}^t a(t,s)g(s,x(\tau(s)))ds \\ &\geq f(t) - \int_0^{T^*} a(t,s)g(s,x(\tau(s)))ds \\ &\geq f(t) - L^*\int_0^{T^*} a(t,s)ds, \quad t \geq T^*, \end{aligned}$$

where $L^* = \sup_{t\in[0,T^*]} |g(t,x(\tau(t)))|$. This inequality, $\limsup_{t\to\infty} f(t) = M > 0$ and $\lim_{t\to\infty} h_{T^*}(t) = 0$ lead a contradiction to $x(t) < 0$.

Example 1. Consider the equation

$$x(t) = f(t) - \int_0^t a(t,s)g(s,x(\tau(s)))ds \qquad (3)$$

where $f(t) = \frac{t}{t^2+1}\{-(t+2\pi)\cos t + \sin t + 2\pi\} + (\frac{t\sin t}{t^2+1})^{\frac{1}{5}}$, $a(t,s) = 0$ if $s > t$, $a(t,s) = \frac{t}{t^2+1}\{(s+2\pi)^2+1\}^2$ for $0 \leq t < \infty$ and $0 \leq s \leq t$,

$\tau(s) = s + 2\pi$ and $g(s, x(\tau(s))) = \{(s+2\pi)^2+1\}^{-1} \times (x(s+2\pi))^5$. Since equation (3) satisfies all conditions of Theorem 1, all solutions of (3) are weakly oscillatory. Such an oscillating solution of (3) is $x(t) = (\frac{t \sin t}{t^2+1})^{\frac{1}{5}}$.

Remark. Example 1 is not examined by the results of [2].

Theorem 2. *Let $\lim_{t\to\infty} f(t) = 0$ or $f(t) \equiv 0$, and $xg(t,x) > 0$ if $x \neq 0$. If the function $h_\sigma(t)$ as in Theorem 1, satisfies $\lim_{t\to\infty} h_\sigma(t) = 0$ for every fixed $\sigma > 0$, then every solution $x(t)$ of (1) is weakly oscillatory or $\lim_{t\to\infty} x(t) = 0$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Suppose that $x(t) > 0$ for $t \geq T > 0$. From (1) and $\tau(t) \geq t$, we have

$$\begin{aligned} 0 < x(t) &= f(t) - \int_0^T a(t,s)g(s,x(\tau(s)))ds \\ &\quad - \int_T^t a(t,s)g(s,x(\tau(s)))ds \\ &\leq f(t) - \int_0^T a(t,s)g(s,x(\tau(s)))ds \qquad (4) \\ &\leq f(t) + L\int_0^T a(t,s)ds, \quad T \leq t < \infty, \\ &\quad \text{where } L = \sup_{t\in[0,T]} g(t,x(\tau(t))). \end{aligned}$$

Since $\lim_{t\to\infty} h_T(t) = \lim_{t\to\infty} \int_0^T a(t,s)ds = 0$, $\lim_{t\to\infty} f(t) = 0$ and (4), we obtain $\lim_{t\to\infty} x(t) = 0$. Let $x(t) < 0$ for $t \geq T^* > 0$. So

$$\begin{aligned} 0 > x(t) &\geq f(t) - \int_0^{T^*} a(t,s)g(s,x(\tau(s)))ds \\ &\geq f(t) - L^*\int_0^{T^*} a(t,s)ds, \qquad (5) \\ &\quad \text{where } L^* = \sup_{t\in[0,T^*]} g(t,x(\tau(t))). \end{aligned}$$

From (5), we have $\lim_{t\to\infty} x(t) = 0$.

Example 2. Consider the integral equation

$$x(t) = f(t) - \int_0^t a(t,s)g(s,x(s))ds \qquad (6)$$

where $f(t) = \frac{t}{\sqrt{t^3+1}} + e^{-t}$, $\quad a(t,s) = 0$ if $s > t$,
$a(t,s) = \frac{e^{2s}}{\sqrt{t^3+1}}$ for $0 \leq t < \infty$ and $0 \leq s \leq t$,
and $g(s,x(s)) = e^s(x(s))^3$.

For equation (6), all conditions of Theorem 2 are satisfied, so that every nonoscillatory solution $x(t)$ of (6) satisfies $\lim_{t\to\infty} x(t) = 0$. Such a solution of (6) is $x(t) = e^{-t}$.

Remark. Theorem 2 is concerned with [2, Theorem 3.6]. But we treat with the case that the existence of $\lim_{t\to\infty} x(t)$ is not assumed.

We note that these theorems 1 and 2 are extended at once to the more general integral equation of Volterra type (Cf. [3],[4])

$$x(t) = f(t) - \sum_{i=1}^{n} \int_0^t a_i(t,s) g_i(s, x(\tau_i(s)))ds, \quad t \geq 0 \tag{7}$$

In (7), $f : [0,\infty) \to R$ is continuous, $g_i : [0,\infty) \times R \to R$ is continuous for every i, $1 \leq i \leq n$ and $a_i : [0,\infty) \times [0,\infty) \to R$ is such that $a_i(t,s) = 0$ if $s > t$, $a_i(t,s) > 0$ for $0 \leq t < \infty$ and $0 \leq s \leq t$, for every i, $1 \leq i \leq n$. Let $a_i(t,s)$ be continuous for $0 \leq t < \infty$ and $0 \leq s \leq t$ for every i, $1 \leq i \leq n$, and $\tau_i(t)$ be continuous, nondecreasing and $\tau_i(t) \geq t$ on $[0,\infty)$, for every i, $1 \leq i \leq n$.

Theorem 3. *Let* $\limsup_{t\to\infty} f(t) = M > 0$ *and* $\liminf_{t\to\infty} f(t) = N < 0$, *where M and N are constants. Let $xg_i(t,x) > 0$ if $x \neq 0$ for every i, $1 \leq i \leq n$, and the function $i - h_\sigma(t) = \int_0^\sigma a_i(t,s)ds$ satisfies $\lim_{t\to\infty} i - h_\sigma(t) = 0$ for every fixed $\sigma > 0$ and i, $1 \leq i \leq n$. Then all solutions of (7) are weakly oscillatory.*

The proof is proceeded as same as in Theorem 1.

References

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