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NEGATIVE DEFINITE KERNELS OF INFINITELY MANY VARIABLES

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In this paper we shall introduce negative definite kernel of infinitely many variables. After giving some equivalent formulation of the notion of negative difiniteness, elementary properties are discussed. We shall also illustrate the relation between negative definiteness and positive definiteness.

1. Properties of Negative-Definite Kernels

The fundamental connection between positive and negative definite kernels was introduced by Schoenberg(1938) [4].

A study of positive and negative definite kernels with invariance properties under a group action may be found in Parthasarathy and Schmidt (1972) [3].

Let X be a nonempty set. A function $\Psi : X \times X \to \mathbf{C}$ is called a negative definite kernel if and only if it is hermitian (i.e. $\Psi(y,x) = \overline{\Psi(x,y)}$ for all $x, y \in X$) and

$$\sum_{j,k=1}^{n} C_j \bar{C}_k \Psi(x_j, x_k) \leq 0 \tag{1.1}$$

for all $n > 2$, $\{x_1, \cdots, x_n\} \subseteq X$ and

$$\{C_1, \cdots, C_n\} \subseteq \mathbf{C} \text{ with } \sum_{j=1}^{n} C_j = 0 \text{ [2].}$$

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If the above inequalities are strict whenever $x_1, \cdots, x_n$ are different and at least one of the $C_1, \cdots, C_n$ does not vanish, then the kernel Ψ is strictly negative definite.

We now list some simple properties of negative definite kernels.

(1) A real-valued kernel Ψ on $X \times X$ is negative definite if and only if Ψ is symmetric (i.e. $\Psi(x,y) = \Psi(y,x)$ for all $x, y \in X$) and

$$\sum_{j,k=1}^{n} C_j C_k \Psi(x_j, x_k) \leq 0$$

for all $n \in N$, $\{x_1, \cdots, x_n\} \subseteq X$ and $\{C_1, \cdots, C_n\} \subseteq R$ and

$$\sum_{j=1}^{n} C_j = 0.$$

(2) The negative definite kernel Ψ satisfies the inequality

$$\Psi(x,x) + \Psi(y,y) \leq 2Re\Psi(x,y)$$

Now we consider a negative definite kernel with an infinite number of variables.

Let $R_\ell^\infty = (-\ell, \ell) \times R^\infty \subset R^1 \times R^\infty = R^\infty (0 < \ell \leq \infty)$ be a layer of the space R^∞, and let $X = (X_1, X')(X_1 \in (-\ell, \ell),\ x^1 = (x_2, x_3, \cdots) \in R^\infty)$ be points of this layer [1]. In R^∞ let us introduce a measure $d\theta(x) = (p(x_1)dx_1) \otimes\ (p(x_2)dx_2) \otimes \cdots$ where $C^1(R^1) \ni p(t) > 0$ is a fixed weight $(\int_{R^1} p(t)dt = 1)$.

In particular, this measure is defined also on any layer R_ℓ^∞, with $\theta(R_\ell^\infty) > 0$.

A kernel $\Psi(x,y), (x, y \in R_{2\ell}^\infty)$ is said to be negative definite if for any cylindrical function $u(x) = u_c(x_1, \cdots, x_m)(u_c \in C_0^\infty(R_\ell^m))$. We have the inequality

$$\int_{R_\ell^\infty} \int_{R_\ell^\infty} \Psi(x,y) u(x) \overline{u(y)} d\theta(x) d\theta(y) \leq 0 \tag{1.2}$$

with $\int_{R_\ell^\infty} u(\cdot) d\theta(\cdot) = 0$.

A similar definition holds in the case $\ell = \infty$, in which case $R_\ell^\infty = R^\infty$, the measure $d\theta(x)$ must be taken to be Gaussian measure.

Lemma 1.1. *If $f : R_{2\ell}^\infty \to \mathbf{C}$ is an arbitrary function, then the kernel $\Psi(x,y) = f(x) + \overline{f(y)}$ is negative definite.*

Proof. Let $\int_{R_\ell^\infty} u(x)d\theta(x) = 0$, from the inequality

$$\begin{aligned}
&\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
&= \int_{R_\ell^\infty}\int_{R_\ell^\infty} [f(x)+\overline{f(y)}]u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
&= \int_{R_\ell^\infty} f(x)u(x)d\theta(x)\int_{R_\ell^\infty}\overline{u(y)}d\theta(y) + \int_{R_\ell^\infty}\overline{f(y)}\,\overline{u(y)}d\theta(y).
\end{aligned}$$

$$\int_{R_\ell^\infty} u(x)d\theta(x) = 0.$$

Hence, Ψ is negative definite.

Lemma 1.2. *The kernel $\Psi(x,y) = (x-y)^2$ on $R_{2\ell}^\infty$ is negative definite, $\int_{R_\ell^\infty} u(x)d\theta(x) = 0$.*

Proof.

$$\begin{aligned}
&\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
&= \int_{R_\ell^\infty}\int_{R_\ell^\infty} (x-y)^2u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
&= \int_{R_\ell^\infty} x^2u(x)d\theta(x)\int_{R_\ell^\infty}\overline{u(y)}d\theta(y) + \int_{R_\ell^\infty} y^2\overline{u(y)}d\theta(y)\int_{R_\ell^\infty} u(x)d\theta(x) \\
&\quad -2\int_{R_\ell^\infty}\int_{R_\ell^\infty} xyu(x)u(y)u(y)d\theta(x)d\theta(y) \le 0
\end{aligned}$$

So Ψ is negative definite.

2. Relation between Positive and Negative Definite Kernels

Lemma 2.1. *Let $\Psi : R_{2\ell}^\infty \to \mathbf{C}$ be a hermitian kernel. Set $X_0 \in R_\ell^\infty$ and put $\Phi(x,y) = \Psi(x,x_0) + \overline{\Psi(y,x_0)} - \Psi(x,y) - \Psi(x_0,x_0)$. Then Φ is positive definite if and only if Ψ is negative definite. If $\Psi(x_0,x_0) > 0$ and $\Phi_0(x,y) = \Psi(x,x_0) - \overline{\Psi(y,x_0)} - \Psi(x,y)$, then Φ_0 is positive definite if and only if Ψ is negative definite.*

Proof. Let Φ be positive definite and let $\int_{R_\ell^\infty} u(\cdot)d\theta(.) = 0$, then

$$0 \le \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Phi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y)$$

$$\begin{aligned}
= & \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,x_0)u(x)\overline{u(x_0)}d\theta(x)d\theta(x_0) \\
& + \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x_0,y)u(x_0)\overline{u(y)}dx(x_0)d\theta(y) \\
& - \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
& - \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x_0,x_0)u(x_0)\overline{u(x_0)}d\theta(x_0)d\theta(x_0).
\end{aligned}$$

Consequently,

$$0 \le -\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y).$$

Hence,

$$\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \le 0.$$

So Ψ is negative definite.

Conversely: Let Ψ be negative definite.

$$\begin{aligned}
0 \ \ge & \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \\
= & \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x,x_0)u(x)\overline{u(x_0)}d\theta(x)d\theta(x_0) \\
& + \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x_0,y)u(x_0)\overline{u(y)}d\theta(x_0)d\theta(y) \\
& - \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi(x_0,x_0)u(x_0)\overline{u(x_0)}d\theta(x_0)d\theta(x_0) \\
& - \int_{R_\ell^\infty}\int_{R_\ell^\infty} \Phi(x,y)u(y)\overline{u(y)}d\theta(x)d\theta(y)
\end{aligned}$$

$$0 \ge -\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Phi(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y)$$

then

$$\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Phi(x,y)u(x)u(y)d\theta(x)d\theta(y) \ge 0$$

i.e. Φ is positive definite.

Now if $\Psi(x_0,x_0) \ge 0$. Then

$$\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Phi_0(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y)$$

$$= \int_{R_\ell^\infty} \int_{R_\ell^\infty} [\Phi(x,y) + \Psi(x_0,x_0)] u(x)\overline{u(y)} d\theta(x) d\theta(y)$$
$$= \int_{R_\ell^\infty} \int_{R_\ell^\infty} \Phi(x,y) u(x)\overline{u(y)} d\theta(x) d\theta(y)$$
$$+ \int_{R_\ell^\infty} \int_{R_\ell^\infty} \Psi(x_0,x_0) u(x_0)\overline{u(x_0)} d\theta(x_0) d\theta(x_0)$$

is positive definite.

Theorem 2.1. *Let $\Psi : R_{2\ell}^\infty \to \mathbf{C}$ be a kernel then Ψ is negative definite if and only if* $\exp(-t\Psi)$ *is positive definite for all $t > 0$.*

Proof. If $\exp(-t\Psi)$ is positive definite then $1 - \exp(-t\Psi)$ is, of course, negative definite and so there for the point wise limit

$$\Psi = \lim_{0<t\to 0} \frac{1}{t}(1 - \exp(-t\Psi)).$$

Now suppose that Ψ is negative definite. We need to obtain that $\exp(-t\Psi)$ is positive definite for $t = 1$. We choose $x_0 \in R_\ell^\infty$ and with Φ as in the Lemma 2.1 we have:

$$-\Psi(x,y) = \Phi(x,y) - \Psi(x,x_0) - \overline{\Psi(y,x_0)} + \Psi(x_0,x_0),$$

where Φ is positive definite. Hence

$$\int_{R_\ell^\infty} \int_{R_\ell^\infty} \exp(-\Psi(x,y)) u(x)\overline{u(y)} d\theta(x) d\theta(y)$$
$$= \int_{R_\ell^\infty} \int_{R_\ell^\infty} \exp[\Phi(x,y) - \Psi(x,x_0) - \overline{\Psi(y,x_0)} + \Psi(x_0,x_0)]$$
$$\cdot \exp[\Psi(x,x_0)u(x)] \cdot \exp[\overline{\Psi(y,x_0)}\,\overline{u(y)}] d\theta(x) d\theta(y)$$
$$= \int_{R_\ell^\infty} \int_{R_\ell^\infty} \exp[\phi(x,y) + \Psi(x_0,x_0)] u(x)\overline{u(y)} d\theta(x) d\theta(y)$$

from the above lemma

$$= \int_{R_\ell^\infty} \int_{R_\ell^\infty} \exp(\Phi_0(x,y)) u(x)\overline{u(y)} d\theta(x) d\theta(y)$$

since $\exp(\Phi_0(x,y))$ is positive definite if $\Phi_0(x,y)$ is positive definite [see 2 page 70]. Hence

$$\int_{R_\ell^\infty} \int_{R_\ell^\infty} \exp(-\Psi(x,y)) u(x) u(y) d\theta(x) d\theta(y) \geq 0.$$

So $\exp(-\Psi)$ is positive definite. Since $t > 0$ and Ψ is negative definite we have $t\Psi$ is also negative definite.

Lemma 2.2. *Let $\Psi(x, y)$ be a negative definite kernel with strictly positive real part. Then $\frac{1}{\Psi}$ is positive definite.*

Proof. Since Ψ is negative definite, then by theorem 3.1 the function $\exp(-t\Psi)$ is positive definite for all $t > 0$. We can write for $(x, y) \in R^{\infty}_{2\ell}$

$$\frac{1}{\Psi(x,y)} = \int_0^{\infty} \exp(-t\Psi)dt.$$

Now

$$\begin{aligned}
&\int_{R^{\infty}_{\ell}} \int_{R^{\infty}_{\ell}} \frac{1}{\Psi(x,y)} u(x)\overline{u(y)} d\theta(x) d\theta(y) \\
&= \int_{R^{\infty}_{\ell}} \int_{R^{\infty}_{\ell}} (\int_0^{\infty} \exp[-t\Psi(x,y)]dt) u(x)\overline{u(y)} d\theta(x) d\theta(y) \\
&= \int_0^{\infty} [\int_{R^{\infty}_{\ell}} \int_{R^{\infty}_{\ell}} \exp[-t\Psi(x,y)] u(x)\overline{u(y)} d\theta(x) d\theta(y)] dt \geq 0
\end{aligned}$$

So $\frac{1}{\Psi}$ is positive definite.

Theorem 2.2. *Let μ be a probability measure on $R^{\infty}_{2\ell}$ such that $0 < \int_0^{\infty} S d\mu(s) < \infty$, and let $\mathbf{L}_{\mu}$ denote its Laplace transform, i.e. $\mathbf{L}_{\mu}(z) = \int_0^{\infty} e^{-sz} d\mu(s), z \in \mathbf{C}$. Then $\Psi : R^{\infty}_{2\ell} \to \mathbf{C}$ is negative definite if and only if $\mathbf{L}_{\mu}(t\Psi)$ is positive definite for all $t \geq 0$.*

Proof. If Ψ is negative definite then by Theorem (2.1) $\exp(-t\Psi)$ is positive definite for all $t > 0$. We have

$$\mathbf{L}_{\mu}(t\Psi) = \int_0^{\infty} \exp(-ts\Psi) d\mu(s)$$

point wise on $R^{\infty}_{2\ell}$, which is positive definite.

If on the other hand $\mathbf{L}_{\mu}(t\Psi)$ is positive definite for all $t > 0$, then for each $(x, y) \in R^{\infty}_{2\ell}$ we get

$$\begin{aligned}
\frac{1}{t}|1 - \mathbf{L}_{\mu}(t\Psi(x,y))| &= \int_0^{\infty} \frac{1 - \exp|-tS\Psi(x,y)|}{t} d\mu(s) \\
&\to \Psi(x,y) \int_0^{\infty} s d\mu(s) \text{ for } t \to 0.
\end{aligned}$$

Being a pointwise limit of negative definite kernels, Ψ itself is negative definite, too.

Definition 2.1. A positive definite kernel ϕ is called infinitely divisible if for each $n \in N$ there exists a positive definite kernel ϕ_n such that $\phi = (\phi_n)^n$.

If Ψ is negative definite then $\phi = e^{-\Psi}$ is infinitely divisible since $\phi_n = \exp(-(\frac{1}{n})\Psi)$ is positive definite and $(\phi_n)^n = \phi$.

Lemma 2.2. *If $f : R_\ell^\infty \to \mathbf{C}$ satisfies $Ref \geq 0$ then for each $\alpha \in [1,2]$ the kernel*

$$\Psi_\alpha(x,y) = -(f(x) + \overline{f(y)})^\alpha$$

is negative definite.

Proof. An equivalent formulation is that the kernel $-(x+\bar{y})^\alpha$ is negative definite on C. This is clear when $\alpha = 1$ and $\alpha = 2$ see Lemma (1.2). For if $\int_{R_\ell^\infty} u(x)d\theta(x) = 0$

$$\int_{R_\ell^\infty}\int_{R_\ell^\infty} \Psi_\alpha(x,y)u(x)\overline{u(y)}d\theta(x)d\theta(y) \leq 0.$$

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