A WEAK PROJECTIVE COVER OF A MODULE

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1. Introduction

In [5], dualizing the notion of an injective envelope, Rotman defined a projective cover of a module and showed it is equivalent to the concept of already well-knowned one.

In [4], the first author showed that a well-knowned projective cover of a module implies the one in a sense of Rotman, but its converse is not always true.

In this paper, we introduce the concept of a weak projective cover of a module, which is same as a projective cover in a sense of Rotman. We have to investigate some properties of weak projective cover and find conditions under which two concepts are equivalent.

Throughout this paper, $R$ denotes a ring with 1 and every module is a unitary left $R$-module. For terminology and notation, we refer to [3], [5].

2. Main results

We define a weak projective cover of a module, which is the dual concept of an injective envelope.

Definition. An epimorphism $\varepsilon : P \rightarrow M$ is a weak projective cover of a module $M$ if $P$ is a projective module and there exists an epimorphism dashed arrow below

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whenever $Q$ is a projective module and $\psi : Q \to M$ is an epimorphism.

Remark. In [4], the first author showed that every projective cover of a module is a weak projective cover, but its converse is not always true. For example, let $\varepsilon : \mathbb{Z} \to \mathbb{Z}_2$ be the natural epimorphism as $\mathbb{Z}$-modules. Then it is not a projective cover but a weak projective cover of $\mathbb{Z}_2$.

**Proposition 1.** Let $\xi : Q \to P$ be a weak projective cover of a projective module $P$ and $\varepsilon : P \to M$ a homomorphism. Then $\varepsilon : P \to M$ is a weak projective cover if and only if $\varepsilon \xi : Q \to M$ is a weak projective cover.

**Proof.** Consider the following diagram

```
      M
     / \    |
    /   \   |
Q  / \  \  |  P
   /  \  \  |
  /   \  \  |
  S    φ   φ
```

where $S$ is a projective module and $\psi : S \to M$ is an epimorphism.

Suppose that $\varepsilon : P \to M$ is weak projective cover. Then $\varepsilon \xi$ is an epimorphism and there exists an epimorphism $\phi : S \to P$ with $\varepsilon \phi = \psi$. Since $\varepsilon : Q \to P$ is a weak projective cover of $P$, there is an epimorphism $\tilde{\phi} : S \to Q$ with $\xi \tilde{\phi} = \phi$. Hence $\varepsilon \xi : Q \to M$ is a weak projective cover of $M$. Conversely, let $\varepsilon \xi : Q \to M$ be a weak projective cover of $M$. Then $\varepsilon$ is an epimorphism and there exists an epimorphism $\tilde{\phi} : S \to Q$ with
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$(\varepsilon \xi) \tilde{\phi} = \psi$. Let $\phi = \xi \tilde{\phi}$. Then $\phi$ is epic and $\varepsilon \phi = \psi$. Hence $\varepsilon : P \to M$ is a weak projective cover.

**Proposition 2.** Let $\varepsilon : P \to M$ be a weak projective cover of $M$ and $\xi : M \to N$ a superfluous epimorphism. Then $\xi \varepsilon : P \to N$ is a weak projective cover of $N$.

**Proof.** Let $Q$ be a projective module and $\psi : Q \to N$ an epimorphism. Then there is an homomorphism $\tilde{\psi} : Q \to M$ with $\xi \tilde{\psi} = \psi$. Moreover, $M = \ker \xi + \im \tilde{\psi}$. Since $\xi$ is superfluous, $M = \im \tilde{\psi}$. Hence $\tilde{\psi}$ is epic. By assumption, there exists an epimorphism $\phi : Q \to P$ such that $\varepsilon \phi = \tilde{\psi}$. It follows that $\xi \varepsilon : P \to N$ is a weak projective cover.

**Theorem 3.** Let $R$ be a IBN (=invariant basis number) ring such that every projective $R$-module is free. If a module $M$ has a weak projective cover, then it is unique up to isomorphism.

**Proof.** Let $\varepsilon : P \to M$ and $\xi : Q \to M$ be two weak projective covers of $M$. Then there are epimorphisms $\phi : P \to Q$ and $\psi : Q \to P$ such that $\xi \phi = \varepsilon$ and $\varepsilon \psi = \xi$. Let $X$ and $Y$ be bases of $P$ and $Q$, respectively. Since $\phi$ is epic, $\phi(X)$ generates $Q$, and hence $|Y| \leq |\phi(X)| \leq |X|$. Similarly, $|X| \leq |Y|$. Thus $|X| = |Y|$. Since $R$ is IBN, $P$ and $Q$ are isomorphic.

**Remark.** It is well-known that a projective cover of a module is unique. However, a weak projective cover of a module need not be unique in general.

**Corollary 4.** Let $R$ be a commutative ring such that every projective $R$-module is free. Then a module $M$ has a unique weak projective cover if it has one.

**Example.** Let $R_1$ be a quasi-local ring, $R_2$ a P.I.D., $R_3$ a Bézout ring, and $R_4 = K[x_1, x_2, \ldots, x_n]$, where $K$ is a field. Then every $R_i$-module, $i = 1, 2, 3, 4$ has the unique projective cover if it has one.

**Theorem 5.** $R$ be a IBN ring such that every projective $R$-module is free. If $M$ has a projective cover and $\varepsilon : P \to M$ is a weak projective cover of $M$, then it is the projective cover.

**Proof.** Let $\xi : Q \to M$ be a projective cover of $M$. Then it is also a weak projective cover. By Theorem 3, there is an isomorphism $\phi : P \to Q$ with $\xi \phi = \varepsilon$. We claim that $\ker \varepsilon$ is superfluous in $P$. Let $N$ be a submodule of $P$ such that $\ker \varepsilon + N = P$. Since $\phi$ is an isomorphism, it follows that $Q = \ker \xi + \phi(N)$. Hence $Q = \phi(N)$, that is, $N = P$. Thus $\ker \varepsilon$ is
Corollary 6. Let $R$ be a left perfect IBN ring such that every projective $R$-module is free. If $\varepsilon : P \to M$ is a weak projective cover of $M$, then it is the projective cover.

Remark. This may be false without the hypothesis of left perfectness. For example, the natural map $\varepsilon : \mathbb{Z} \to \mathbb{Z}_2$ is a weak projective cover of $\mathbb{Z}_2$, but it not the projective cover.

Corollary 7. Let $R$ be a left perfect IBN ring such that every projective $R$-module is free. Then a direct sum of any weak projective covers is also a weak projective cover of the direct sum of modules.

Remark. In general, a direct sum of weak projective covers is not a weak projective cover of the direct sum of modules.

For example, let $\xi : \mathbb{Z} \to \mathbb{Z}_3$ be the natural epimorphism. We show that it is weak projective cover of $\mathbb{Z}_3$. Consider the following diagram.

\[
\begin{array}{cccc}
0 & \to & \mathbb{Z} & \xleftarrow{\xi} & \mathbb{Z}_3 & \to & 0 \\
& & \downarrow{\phi} & & \downarrow{\psi} & & \\
& & \mathbb{Z}_3 & & Q & & \\
\end{array}
\]

where $Q$ is a projective module over $\mathbb{Z}$ and $\psi$ an epimorphism. Since $Q$ is projective, we may assume that $Q = \prod_{\alpha} \mathbb{Z}_3$, where $\mathbb{Z}_3 = \mathbb{Z}$ for each $\alpha$. Let $u_\alpha : \mathbb{Z}_3 \to \prod_{\alpha} \mathbb{Z}_3$ be the $\alpha$th injection. Since $\psi$ is epic, there exists $\alpha$ such that $\psi u_\alpha = \xi$ or there is $\beta$ such that $\psi u_\beta = 2\xi$. For each $\alpha$, define $\phi_\alpha : \mathbb{Z}_3 \to \mathbb{Z}$ as follows:

$$
\phi_\alpha(1) = \begin{cases} 
1 & \text{if } \psi u_\alpha = \xi \\
-1 & \text{if } \psi u_\alpha = 2\xi \\
0 & \text{if } \psi u_\alpha = 0.
\end{cases}
$$

Then $\xi \phi_\alpha = \psi u_\alpha$ for each $\alpha$. Let $\phi$ be the coproduct map of the family $\{\phi_\alpha\}$. Then the existence of $\alpha$ or $\beta$ implies $\phi$ is an epimorphism. Moreover $\xi \phi u_\alpha = \xi \phi_\alpha = \psi u_\alpha$ for each $\alpha$. We have thus $\xi \phi = \psi$. So $\xi : \mathbb{Z} \to \mathbb{Z}_3$ is a weak projective cover of $\mathbb{Z}_3$. We already showed that $\varepsilon : \mathbb{Z} \to \mathbb{Z}_2$
is a weak projective cover of $\mathbb{Z}_2$. However, $\varepsilon \times \xi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ is not a weak projective cover of $\mathbb{Z}_2 \times \mathbb{Z}_3$. Indeed, let $\psi : \mathbb{Z} \to \mathbb{Z}_2 \times \mathbb{Z}_3$ be defined by $\psi(1) = (1, 1)$. Then it is an epimorphism. But there exist no epimorphisms from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$.

References


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