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ON SUBSOCLES OF S_2-MODULES II

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Introduction

In recent years a new theory for a special module called S_2-module, has been developed and the well known results of torsion abelian groups have been shown to be valid for this module (see [1,2,6,7,8,9,10,11,12]). In [14], a submodule N of an S_2-module M is called h-pure if $N \cap H_n(M) = H_n(N)$ for all $n > 0$. It is very natural to consider the case when $N \cap H_n(M) \subseteq H_k(N)$, where n and k are related by some rules. In this connection, J. D. Moore [3] got a useful technique and introduced the concept of imbedded subgroups of primary abelian groups. The main purpose of this paper is to generalize the concept of h-purity of submodules and to make a rigourous study of this concept and their consequences. This paper is in the continuation of [6].

The paper consists of four sections. In Section 1, we state preliminary results needed for subsequent sections. Section 2 deals with a special type of imbedded submodule called as 'regularly imbedded submodule'. We have shown that an important result of P. Hill and C. Megibben [13, Theorem 1] holds for this module, namely, "An h-neat submodule of an S_2-module M supported by an h-dense subsocle of M is h-pure and h-dense in M" (Corollary 2.3). In Section 3, we study ℓ-quasi-complete modules and obtain a characterization for this (Theorem 3.14). In Section 4, we introduce the concept of minimal ℓ-imbedding and obtain different characterizations for its existence in an S_2-module with certain property.

1. Preliminaries

The notations and terminology have been adopted from [2,6,11,12]. As done by J. D. Moore [3] for groups we define an ℓ-imbedded submodule as :

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A submodule N of an S_2-module M is called ℓ-imbedded if there exists a non-decreasing function $\ell : Z^+ \to Z^+$ such that $N \cap H_{\ell(n)}(M) \subseteq H_n(N)$ for each $n \in Z^+$. Trivially, ℓ-imbedded submodules are h-pure.

Now, we state some basic results whose proofs are trivials.

M will be an S_2-module throughout this section.

Lemma 1.1. *Let $K \subseteq N$ be submodules of M such that K is ℓ-imbedded in M, then K is ℓ-imbedded in N.*

Lemma 1.2. *If K is ℓ_1-imbedded in N and N is ℓ_2-imbedded in M. Then K is $\ell_2 \circ \ell_1$-imbedded in M.*

Lemma 1.3. *Let $K \subseteq N$ be submodules of M such that N is ℓ-imbedded in M, then N/K is ℓ-imbedded in M/K.*

Lemma 1.4. *If N is ℓ-imbedded in M, then it is $\ell' \circ \ell$-imbedded for every ℓ'.*

Lemma 1.5. *Let $K \subseteq N$ be submodules of M such that K is ℓ_1-imbedded in M and N/K is ℓ_2-imbedded in M/K. Then N is $\ell_2 \circ \ell_1$-imbedded in M.*

Lemma 1.6. *Let N_1 and N_2 be submodules of M.*
(a) If $N_1 \cap N_2$ is ℓ-imbedded in N_1, then N_2 is ℓ-imbedded in $N_1 + N_2$.
(b) If $N_1 + N_2$ is ℓ-imbedded in M and $N_1 \cap N_2$ is ℓ-imbedded in N_1, then N_2 is $\ell \circ \ell$-imbedded in M.
(c) If $N_1 + N_2$ and $N_1 \cap N_2$ are ℓ-imbedded in M, then N_1 and N_2 are $\ell \circ \ell$-imbedded in M.
(d) If $N_1 \cap N_2$ is ℓ-imbedded in $N_1 + N_2$, then N_1 and N_2 are $\ell \circ \ell$-imbedded in $N_1 + N_2$.

Corollary 1.7. *If $M/K = N/K \oplus T/K$ such that K is ℓ-imbedded in N, then T is ℓ-imbedded in M.*

Lemma 1.8. *For an ℓ-imbedded submodule N of M, $N \cap M^1 = N^1$.*

Corollary 1.9. *If $N \subseteq M^1$, then N is imbedded in M if and only if N is h-divisible.*

Lemma 1.10. *If N is an imbedded submodule of M, then $\bar{N}$ is imbedded in M if and only if $(M/N)^1$ is h-divisible, where $\bar{N}$ is the closure of N defined as $\bar{N}/N = (M/N)^1$.*

Lemma 1.11. *For a submodule N of M, $\bar{N} = \cap_{n=1}^{\infty}(N + H_n(M))$.*

Lemma 1.12. *A submodule N of M is h-dense in M if and only if $\bar{N} = M$.*

2. Regularly imbedded submodules

A submodule N of an S_2-module M is called regularly imbedded in M with index k, if $N \cap H_{k+n}(M) \subseteq H_n(N \cap H_k(M))$ for every n.

Evidently, if N is regularly imbedded with index k, then $N \cap H_{k+n}(M) \subseteq H_n(N)$ gives that the regularly imbedded submodules are ℓ-imbedded for some $\ell : Z^+ \to Z^+$, therefore, the results of Section 1 can be suitably carried over to regularly imbedded submodules. Moreover, the regularly imbedded submodules of index zero are exactly the h-pure submodules. Also, we can easily prove the following:

Proposition 2.1. *Let N be a regularly imbedded submodule of an S_2-module M with index 1. If N is h-neat in M then it is h-pure.*

We recall from [2] that a subsocle S of an S_2-module M is h-dense in $soc(M)$, if $soc(M) = S + soc(H_n(M))$ for every n.

Now, we prove the following proposition which is a generalizations of a result of J. D. Moore [3, Proposition 3.5].

Proposition 2.2. *Let N be a submodule of an S_2-module M such that $soc(N)$ is h-dense in $soc(M)$. If $N \cap H_{m+1}(M) \subseteq H_1(N)$ for some m, then*

(a) *N is regularly imbedded with index m in M.*

(b) $H_m(M) \subseteq \bar{N}$.

Proof. (a) Firstly, we show that $N \cap H_{m+1}(M) \subseteq H_1(N \cap (soc(M) + H_m(M)))$. For this, let $x \in N \cap H_{m+1}(M)$, be a uniform element then there exists a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR) = m+1$. Also, $x \in H_1(N)$ implies that there exists a uniform element $z \in N$ such that $x \in zR$ and $d(zR/zR) = 1$. Let $wR/xR = soc(yR/xR)$, then $d(wR/xR) = 1$. Hence, appealing to the condition (II) of the S_2-module, we get $e(z - w) \leq d(zR/xR) = 1$, and thus, $z - w \in soc(M)$. Therefore, $z \in soc(M) + H_m(M)$, and so $x \in H_1(N \cap (soc(M) + H_m(M)))$, which proves that $N \cap H_{m+1}(M) \subseteq H_1(N \cap (soc(M) + H_m(M)))$. Now, we prove that $N \cap H_{m+n}(M) \subseteq H_n(N \cap H_m(M))$ for every n. Since, $soc(N)$ is h-dense in $soc(M)$, we have $N \cap (soc(M) + H_m(M)) \subseteq N \cap (soc(N) + H_m(M))$ $= soc(N) + N \cap H_m(M)$ so, $N \cap H_{m+1}(M) \subseteq H_1(soc(N) + N \cap H_m(M))$ $= H_1(N \cap H_m(M))$. Let us assume that $N \cap H_{m+n}(M) \subseteq H_n(N \cap H_m(M))$

for some n. Then, as done above, it is easy to show that

$$\begin{aligned} N\cap H_{m+n+1}(M) &\subseteq H_1(N\cap(soc(M)+H_{m+n}(M))) \\ &\subseteq H_1(soc(N)+N\cap H_{m+n}(M)) \\ &= H_1(N\cap H_{m+n}(M)) \\ &\subseteq H_{n+1}(N\cap H_m(M)), \text{ by assumption.} \end{aligned}$$

Hence, the result follows by induction and N is regularly imbedded with index m.

(b) If $x \in soc(H_m(M))$ is a uniform element, then trivially, using Lemma 1.11, $x \in \bar{N}$. Let us assume that all the uniform elements of $H_m(M)$ of exponent at most k belong to $\bar{N}$ and let $y \in H_m(M)$ be a uniform element of exponent $k+1$. Then there exists a uniform element $z \in H_m(M)$ such that $z \in yR$ and $d(yR/zR) = 1$. So $z \in H_{m+1}(M)$. Also, $e(z) = k$, so $z \in \bar{N}$, by assumption. So that $z = u + t$, where $u \in N$ and $t \in H_{m+n+1}(M)$ for every n. Then $u = z - t \in H_{m+1}(M)$ and so $u \in N \cap H_{m+1}(M) \subseteq H_1(N \cap H_m(M))$, by (a). Therefore, $z \in H_1(N \cap H_m(M) + H_{m+n}(M))$. Hence, there exists a uniform element $w \in (N \cap H_m(M) + H_{m+n}(M))$ such that $z \in wR$ and $d(wR/zR) = 1$. Hence, appealing to the condition (II), we get $e(y-w) \leq d(yR/zR) = 1$. i.e. $y - w \in soc(M)$. Thus, $y \in soc(M) + N + M_{m+n}(M)$ for every n. Hence, $y \in \bar{N}$, as $soc(N)$ is h-dense in $soc(M)$. The result follows by induction.

We recall from [2] that an S_2-module M is called an S_3-module if it further satisfies one more conditions:

(III) For every finitely generated submodule N of M, $R/ann(N)$ is right Artinian.

In [2], we have generalized a result of P. Hill and C. Megibben [15, Theorem 1] for S_3-modules. As an application of the Proposition 2.2, we further improve that for S_2-modules as follows:

Corollary 2.3. *An h-neat submodule N of an S_2-module M supported by an h-dense subsocle of M is h-pure and h-dense in M.*

3. Subsocles and ℓ-quasi-completeness

In [6] we have made an study of quasi-complete S_2-modules. Here we introduce ℓ-quasi-complete S_2-modules and get a characterization.

Definition 3.1. An S_2-module M is called ℓ-quasi- complete if the closure

$\bar{N}$ of every ℓ-imbedded submodule N of M is an imbedded submodule of M.

As it is remarked earlier that ℓ-imbedded submodules are exactly the h-pure submodules, hence quasi-complete S_2-modules are ℓ-quasi-complete. Apparently, ℓ-quasi-complete S_2-modules do not seem to be quasi-complete, but the following proposition shows that ℓ-quasi-complete S_2-modules are quasi-complete. The proof, being analogous to [3, Proposition 2.9], is omitted.

Proposition 3.2. *Let N be an ℓ-imbedded submodule of an S_2-module M. If $\bar{N}$ is imbedded in M, then $\bar{N}$ is ℓ-imbedded.*

As defined in [6], an S_2-module M is called separable if it has no uniform element of infinite height. Also, an S_2-module M is reduced if 0 is its only h-divisible submodule.

The following proposition can be proved easily.

Proposition 3.3. *A reduced ℓ-quasi-complete S_2-module is separable.*

Definition 3.4. An ℓ-imbedded submodule of an S_2-module M is said to be strongly ℓ-imbedded if every subsocle S of M containing $soc(N)$ supports an ℓ-imbedded submodule of M containing N.

Proposition 3.5. *Every ℓ-imbedded submodule of an ℓ-quasi-complete S_2-module is strongly ℓ-imbedded.*

Proof. Let M be an ℓ-quasi-complete S_2-module and for any ℓ-imbedded submodule N and for any subsocle S of M containing $soc(N)$, let $\mathcal{F} = \{H \subseteq M | H \text{ is } \ell\text{-imbedded in } M, N \subseteq H \text{ and } soc(H) \subseteq S\}$. Then we can find a maximal ℓ-imbedded submodule K of M containing N such that $soc(K) \subseteq S$. We assert that $soc(K) = S$. Suppose on contrary that there exists a uniform element $x \in S$ such that $x \notin K$. Then $\bar{x} = x + K$ is uniform element of $(S + K)/K$. There are two cases:

Case I: If $H_{M/K}(\bar{x}) = n < \infty$. Then we choose a uniform $y \in M$ such that $x \in yR$ and $d(\bar{y}R/\bar{x}R) = n$. Using [14, Lemma 1], $\bar{y}R$ is a summand of M/K, hence, is ℓ-imbedded in M/K-consequently, by Lemma 1.5, $yR+K$ is ℓ-imbedded in M. Trivially, $soc(yR) = xR$ and $xR \cap K = 0$, therefore, $yR \cap K = 0$, and so $yR \oplus K$ is ℓ-imbedded in M such that $soc(yR \oplus K) \subseteq S$, which contradicts the maximality of K.

Case II: If $H_{M/K}(\bar{x}) = \infty$, then as M is ℓ-quasi-complete, we have, by Lemma 1.10, $(M/K)^1$ to be h-divisible. Therefore, by [12, Theorem

3], $(M/K)^1 = \oplus \sum \bar{U}_i$, where each $\bar{U}_i$ is a uniform submodule of infinite length and $soc(\bar{U}_j) = \bar{x}R$ for some j. We write $\bar{U}_j = T/K$. Now, let $z \in soc(T)$ be a uniform element with $z \notin K$, then $\bar{z} = z + K$ is a uniform element of $(xR + K)/K$. Hence, $z + K = xr + K$ yields that $z \in xR + soc(K)$. Therefore, $soc(T) = xR \oplus soc(K)$. Also, as K is ℓ-imbedded and T/K, being h-divisible, is ℓ-imbedded in M/K, we find that T is ℓ-imbedded in M. Thus, T is ℓ-imbedded in M containing N such that $soc(T) = xR \oplus soc(K) \subseteq S$, which again contradicts the maximality of K. Hence, $soc(K) = S$ and the proposition follows.

Analogous to h-pure-complete modules [1], we call an S_2-module M to be ℓ-imbedded-complete if every subsocle of M supports an ℓ-imbedded submodule of M. Then ℓ-imbedded-complete modules are exactly the h-pure-complete modules.

The following result, analogue to [5, Cor. 74.2], can be easily deduced from Proposition 3.5.

Corollary 3.6. *An ℓ-quasi-complete S_2-module is ℓ-imbedded-complete.*

Also, we have the the following generalization of [6, Proposition 8].

Proposition 3.7. *A reduced ℓ-imbedded-complete S_2-module M is separable.*

Proof. Using the definition of ℓ-imbedded-completeness, we get $soc(M^1) = soc(K)$, for some ℓ-imbedded submodule K of M. Now, for any uniform element $x \in soc(K), x \in K^1$ (using Lemma 1.8). Hence, by [12, Lemma 2], K is h-divisible. Hence $K = 0$ and consequently, $M^1 = 0$.

Proposition 3.5 provides a necessary condition for ℓ-quasi-complete S_2-modules. In order to get a characterization, we prove some results on subsocles.

Proposition 3.8. *Let M be an S_3-module and N be an ℓ-imbedded submodule of M, then $\overline{soc(N)} \cap soc(M) = soc(\bar{N})$.*

Proof. We have

$$\begin{aligned}\overline{soc(N)} \cap soc(M) &\subseteq (soc(N) + H_n(M)) \cap soc(M), \text{ for all } n.\\ &= soc(N) + soc(H_n(M)), \text{ for all } n.\\ &\subseteq soc(N + H_n(M)).\end{aligned}$$

Therefore, $\overline{soc(N)} \cap soc(M) \subseteq soc(\bar{N})$. Now, to show the equality, we need only to show that $soc(N + H_{\ell(n+1)-1}(M)) \subseteq soc(N) + H_n(M)$ for every

n. Let $x \in soc(N + H_{\ell(n+1)-1}(M))$ be a uniform element, then $e(x) = 1$ and $x = y + z$ where $y \in N$ and $z \in H_{\ell(n+1)-1}(M)$. If $ann(xR) = P$, then $yr = -zr$ for every $r \in P$. i.e. $yP = zP$. In case $zP = zR$, we have $yP = zR$. So that for $r_1 \in P$, $z = yr_1$, i.e. $x = y + yr_1 \in N$ and hence $x \in soc(N)$ implies that the assertion follows. Similarly, $yP = yR$ gives that $x \in H_{\ell(n+1)-1}(M) \subseteq H_n(M)$ and the assertion follows. So, we consider the case when $zP < zR$ and $yP < yR$. Now $z \in H_{\ell(n+1)-1}(M)$ gives that $zP \subseteq H_{\ell(n+1)}(M)$. So that $yP \subseteq N \cap H_{\ell(n+1)}(M) \subseteq H_{n+1}(N)$ which yields $yr \in H_{n+1}(N)$, for every $r \in P$. Therefore, there exists a uniform element $t \in N$ such that $yr \in tR$ and $d(tR/yrR) = n + 1$. Let $soc(tR/yrR) = uR/yrR$, then $d(uR/yrR) = 1$. Also, $d(tR/uR) = n$ yields that $u \in H_n(N)$. Moreover, $yP < yR$ implies that there exists a uniform element $v \in yR$ such that $yrR \subseteq vR$ and $d(vR/yrR) = 1$. Hence, appealing to the condition (II), we find that $e(v - u) \leq d(vR/yrR) = 1$ and so, $v - u \in soc(N)$. Now, $v = yr_1, r_1 \in R$, $r_1 \notin P$. Obviously, $xr_1R = xR$, as $e(x) = 1$. So that $x = xr_1r_2, r_2 \in R$, i.e.

$$\begin{aligned} x &= xr_1r_2 \\ &= yr_1r_2 + zr_1r_2 \\ &= vr_2 + zr_1r_2 \\ &= (v-u)r_2 + ur_2 + zr_1r_2 \in soc(N) + H_n(M). \end{aligned}$$

Hence the proposition follows.

Definition 3.9. Let S be a subsocle of an S_2-module M and $\bar{S}$ be the closure of S in M. Then closure of S in $soc(M)$ is given as $\bar{S} \cap soc(M)$. S is called closed in $soc(M)$ if $S = \bar{S} \cap soc(M)$.

Proposition 3.10. *Let M be an S_3-module and N be an ℓ-imbedded submodule of M such that $soc(N)$ is closed in $soc(M)$, then $\bar{N} \cap H_{\ell(1)-1}(M) \subseteq N$.*

Proof. Using the proposition 3.8 and the definition 3.9, we have $soc(\bar{N}) = soc(N)$. So that $soc(\bar{N}) \cap H_{\ell(1)-1}(M) \subseteq N$. Now, let us assume that for every uniform element $x \in \bar{N} \cap H_{\ell(1)-1}(M)$ with $e(x) = k$, $x \in N$. Let y be a uniform element of $\bar{N} \cap H_{\ell(1)-1}(M)$ such that $e(y) = k + 1$. Then there exists a uniform element $z \in \bar{N} \cap H_{\ell(1)-1}(M)$ such that $z \in yR$ and $d(yR/zR) = 1$. Trivially $e(z) = k$. Hence, by assumption, $z \in N$. Also, $y \in H_{\ell(1)-1}(M)$ implies that $z \in H_{\ell(1)}(M)$. So that $z \in N \cap H_{\ell(1)}(M) \subseteq H_1(N)$. Consequently, there exists a uniform element $w \in N$ such that $z \in wR$ and $d(wR/zR) = 1$. Hence appealing to the condition (II), we find that $e(y-u) \leq d(yR/zR) = 1$ i.e. $y - w \in soc(\bar{N})$. So that $y \in N + soc(\bar{N}) = N$. Hence the result follows by induction.

As a consequence of Proposition 3.10, we have

Corollary 3.11. *Let N be an ℓ-imbedded submodule of an S_3-module M such that $soc(N)$ is closed in $Soc(M)$, then $\bar{N}$ is ℓ-imbedded in M.*

Definition 3.12. A submodule N of an $S_{\mathcal{I}}$-module M is called semi-strongly ℓ-imbedded in M if for each subsocle S of M containing $soc(N)$, there exists a subsocle T of M such that

(1) $soc(\bar{N}) \cap S \subseteq T \subseteq S$.

(2) T is h-dense in S.

(3) T supports an ℓ-imbedded submodule of M containing N.

In view of the results 3.8, 3.10, 3.11 and 1.2 the following theorem, a characterization of semi-strongly ℓ-imbedded submodules, can be well adopted from [3, Proposition 3.3].

Theorem 3.13. *An ℓ-imbedded submodule N of an S_3-module M is semi-strongly ℓ-imbedded if and only if $\bar{N}$ is ℓ-imbedded submodule of M.*

Now, let us consider an S_3-module M in which every ℓ-imbedded submodule is strongly ℓ-imbedded. Then for any ℓ-imbedded submodule N of M and for every subsocle S of M containing $soc(N)$, there exists an ℓ-imbedded submodule K of M containing N such that $soc(K) = S$. Taking $T = S$, it follows from definition 3.12 that N is semi-strongly ℓ-imbedded in M. Hence by Theorem 3.13, $\bar{N}$ is ℓ-imbedded submodule of M. Thus, M is ℓ-quasi-complete.

These arguments together with Proposition 3.5, give rise to the following characterization of ℓ-quasi-complete modules.

Theorem 3.14. *An S_3-module M is ℓ-quasi-complete if and only if every ℓ-imbedded submodule of M is strongly ℓ-imbedded.*

4. Minimal ℓ-imbeddings

J. D. Moore [3] introduced the concept of minimal ℓ-imbedding in primary abelian groups and deduced some important results of ℓ-quasi-complete abelian groups. Here we make an analogus study for S_2-modules and generalize some of our own results from [6].

Definition 4.1. Let N be a submodule of an S_2-module M. An ℓ-

imbedded submodule K of M is said to be an ℓ-hull (or minimal ℓ-imbedding) of N in M if K is a minimal ℓ-imbedded submodule of M containing N.

Obviously, if K is ℓ-imbedded, it is ℓ^2-imbedded (where $\ell^2 = \ell \circ \ell$) but the converse is not true. However, as remarked in [3], if K is ℓ-imbedded ℓ^2-hull of N in M, it is an ℓ-hull of N in M.

The following Proposition is a backbone for this section.

Proposition 4.2. *Let N be a submodule of an S_2-module M such that $N \subseteq M^1$. If K is an ℓ-hull of N in M, then K is h-divisible.*

Proof. Suppose on contrary that K is not h-divisible, then by [12, Lemma 2], there exists a uniform element $x \in soc(K)$ such that $H_K(x) = n < \infty$, Hence, by [14, Lemma 1], we can find a bounded summand T of K such that $K = T \oplus L$. Then for any uniform element $u \in N \subseteq K$, we have $u = t + z$, where $t \in T$ and $z \in L$ are uniform elements such that $H(u) = \min\{N(t), H(z)\}$. Since, $H(u) = \infty$, it follows that $t = 0$ i.e. $u = z$. Hence, $N \subseteq L$, where L, being ℓ-imbedded is ℓ-imbedded, by Lemma 1.4. This contradicts the minimality of K. Hence, the proposition follows.

We find a submodule K of an S_2-module M to be an h-divisible hull of a submodule N of M, if K is a minimal h-divisible submodule of M containing N. It follows from Proposition 4.2 that if $N \subseteq M^1$, then an ℓ-hull of N in M is an h-divisible hull. Conversely, if $N \subseteq M^1$, then an h-divisible hull of N in M is easily found to be an ℓ-hull of N in M. Thus, we can extend the above proposition as

Proposition 4.3. *Let N be a submodule of an S_2-module M such that $N \subseteq M^1$. Then a submodule K of M is an ℓ-hull of N in M if and only if K is an h-divisible hull of N in M.*

Definition 4.4. An S_2-module M is said to have I.C.C. (Imbedded Chain Condition) if every descending chain of imbedded submodules of M terminates after a finite number of steps.

Analogous to [6, Proposition 11], we have

Proposition 4.5. *Let M be an S_2-module with I.C.C. and N, K be submodules of M such that K is ℓ-imbedded in M with $K \subseteq N \subseteq \bar{K}$. Then N has an ℓ-imbedded ℓ^2-hull in M if and only if $N \subseteq K_1$, where K_1 is the submodule of M containing K for which K_1/K is the maximal h-divisible submodule of M/K.*

Proof. If T is an ℓ-imbedded ℓ^2-hull of N in M, then, by Lemma 1.3, T/K is an ℓ-imbedded ℓ^2-hull of N/K in M/K. Since, $N/K \subseteq \bar{K}/K = (M/K)^1$, therefore, by Proposition 4.2, T/K is h-divisible submodule of M/K. So that $T \subseteq K_1$ and hence $N \subseteq K_1$. Conversely, if $N \subseteq K_1$, then N/K is contained in K_1/K the maximal h-divisible submodule of M/K containing N/K. If K_1/K is also an h-divisible hull of N/K in M/K, then as $N/K \subseteq \bar{K}/K = (M/K)^1$, we find, by proposition 4.3 that K_1/K is an ℓ-imbedded ℓ-hull of N/K in M/K. Consequently, K_1 is ℓ-imbedded ℓ^2-hull of N in M. If K_1/K is not an h-divisible hull of N/K in M/K, then we get an h-divisible submodule K_2/K of M/K containing N/K and contained in K_1/K. If K_2/K is an h-divisible hull of N/K in M/K, then K_2 is an ℓ-imbedded ℓ^2-hull of N in M. If not, then continuing this process, we get a descending chain of ℓ-imbedded submodules of M containing N which terminates. Hence, we get an ℓ-imbedded ℓ^2-hull of N in M and the proposition follows.

The following theorem gives a characterization for ℓ-quasi-complete modules.

Theorem 4.6. *An S_2-module M with I.C.C. is ℓ-quasi-complete if and only if for all submodules N, K, with K ℓ-imbedded in M and $K \subseteq N \subseteq \bar{K}$, N has an ℓ-imbedded ℓ^2-hull in M.*

Proof. Suppose that M is ℓ-quasi-complete. Then for all submodules, N, K with K ℓ-imbedded in M and $K \subseteq N \subseteq \bar{K}$, we have, by Lemma 1.10, $\bar{K}/K$ to be h-divisible. If $\bar{K}/K$ is a minimal h-divisible submodule of M/K containing N/K, then as $N/K \subseteq \bar{K}/K = (M/K)^1$, using Proposition 4.3, we find that $\bar{K}/K$ is I-imbedded ℓ-hull of N/K in M/K. Consequently, $\bar{K}$ is an ℓ-imbedded ℓ^2-hull of N in M. If $\bar{K}/K$ is not minimal h-divisible containing N/K, then using I.C.C, we can find an ℓ-imbedded ℓ^2-hull of N in M. The proof of the converse part is quite analogoue to that of (3) $\Rightarrow$ (1) of [3, Theorem 3.3].

Towards the end of this section, we have the following theorem, analogous to [6, Theorem 12] which is an other characterization for ℓ-quasi-complete modules. The proof is quite analogous to the above theorem 4.6, hence is omitted.

Theorem 4.7. *An S_2-module M with I.C.C. is ℓ-quasi-complete if and only if every submodule N of M containing an ℓ-imbedded submodule K of M with N/K h-divisible, has an ℓ-imbedded ℓ^2-hull in M.*

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