Rotational Extended Triple Systems

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1. Introduction

An extended triple system of order $v$ is a pair $(V, B)$ where $V$ is a $v$-set and $B$ is a collection of nonordered triples from $V$ (called blocks), where each triple may have repeated elements, such that every pair of elements of $V$, not necessarily distinct, is contained in exactly one block. The blocks of $B$ are of three types:

\{a, a, a\}, \ {b, b, c}, \ {x, y, z},

where the element $a$ is called an idempotent and $b$ a nonidempotent of the system $(V, B)$. We will denote by $E(v; n)$ an extended triple system of order $v$, which has $n$ idempotents. It is easy to see that if there exists an $E(v; n)$, then $0 \leq n \leq v$. Johnson and Mendelsohn [3] obtained a necessary condition for the existence of an $E(v; n)$, and Bennett and Mendelsohn [1] showed that this necessary condition was also sufficient.

Theorem 1.1 [1,3]. There exists an $E(v; n)$ if and only if

(i) $v \equiv 0 \pmod{3}$ and $n \equiv 0 \pmod{3}$ or
(ii) $v \equiv 1$ or $2 \pmod{3}$ and $n \equiv 1 \pmod{3}$ or
(iii) $v$ is even and $n \leq \frac{v}{2}$ or
(iv) $n = v - 1$ and $v = 2$.

An automorphism of an $E(v; n)$ $(V, B)$ is a permutation $\alpha$ of $V$ which fixes $B$ setwise, i.e. $\alpha(B) = B$ where $\alpha(X) = \{\alpha(x) | x \in X\}$.

Let $\alpha$ be a permutation of degree $v$ and type $[\alpha] = [\alpha_1, \alpha_2, \ldots, \alpha_v]$, i.e. the disjoint cycle decomposition of $\alpha$ contains $\alpha_i$ cycles of length $i$ and $\sum i\alpha_i = v$. An $E(v; n)$ admitting $\alpha$ as an automorphism will be denoted by $E_\alpha(v; n)$. If $[\alpha] = [0, 0, \ldots, 0, 1]$, then an $E_\alpha(v; n)$ is called cyclic. If

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[\alpha] = [1, 0, 0, \cdots, 0, k, 0, 0, \cdots, 0]$, i.e. $\alpha_1 = 1$, $\alpha_{(v-1)/k} = k$ and $\alpha_i = 0$ otherwise, then an $E_\alpha(v; n)$ is called $k$-rotational.

A Steiner triple system of order $v$, denoted by $STS(v)$, is a pair $(V, B)$ where $V$ is a $v$-set and $B$ is a set of 3-subsets of $V$, called blocks, such that every 2-subset of $V$ is contained in precisely one block. It is well-known that a $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6), Peltesohn [5] has shown that a cyclic $STS(v)$ exists if and only if $v \equiv 1$ or 3 (mod 6) and $v \neq 9$. The following theorem is from [2, 6].

**Theorem 1.2** [2, 6]. There exists a $k$-rotational $STS(v)$ with $1 \leq k \leq 4$ if and only if

(i) if $k = 1$ then $v \equiv 3$ or 9 (mod 24);

(ii) if $k = 2$ then $v \equiv 1, 3, 7, 9, 15$ or 19 (mod 24);

(iii) if $k = 3$ then $v \equiv 1$ or 19 (mod 24);

(iv) if $k = 4$ then $v \equiv 1, 9, 13$ or 21 (mod 24).

In this paper, we obtain a necessary and sufficient condition for the existence of cyclic $E(v; n)$'s and $k$-rotational $E(v; n)$'s with $k \leq 2$, respectively, and a necessary condition for the existence of 3-rotational $E(v; n)$'s and also show that this necessary condition is sufficient, except possibly for $v \equiv 37$ or 55 (mod 72) and $n = (v + 2)/3$ or $(2v + 1)/3$.

2. Cyclic Extended Triple Systems

If $(V, B)$ is a cyclic $E(v; n)$ with cyclic automorphism $\alpha$, then $B$ can be partitioned into disjoint orbits under $\alpha$, i.e. an orbit of a block $\{a, b, c\}$ under $\alpha$ is the collection of blocks $\{\alpha^i(a), \alpha^i(b), \alpha^i(c)\}$ where $0 \leq i \leq v$. Thus, a collection of blocks taken from each orbit precisely once, called base blocks, represents the whole blocks $B$. The length of a base block is the number of blocks in the orbit containing the base block.

Throughout this section, we will assume that our cyclic $E(v; n)$ is based on $V = Z_v$, the additive group of integers modulo $v$ with residue classes $\{0, 1, \cdots, v - 1\}$ and the corresponding cyclic automorphism is $\alpha = (0 1 \cdots v - 1)$. If $\{a, b, c\}$ is a base block of a cyclic $E(v; n)$, then its length is either $v$ or $v/3$. Thus, if there exists a cyclic $E(v; n)$, then we have either $n = 0$ or $n = v$.

**Theorem 2.1.** A necessary and sufficient condition for the existence of a cyclic $E(v; n)$ is

(i) $n = v$ and $v \equiv 1$ or 3 (mod 6), $v \neq 9$ or
(ii) \( n = 0 \) and \( v \equiv 3 \pmod{6} \).

Proof. \((\Rightarrow)\) (i) It follows from the fact that the class of cyclic \( E(v;v) \)'s is the class of cyclic \( STS(v) \)'s.

(ii) If \( n = 0 \), then \( v \equiv 0 \pmod{3} \) by the existence of an \( E(v;n) \). If \( m \) and \( k \) are the numbers of base blocks of a cyclic \( E(v;0) \) whose lengths are \( v \) and \( v/3 \), respectively, then we have

\[
mv + k(v/3) = \binom{v}{2} + 2v/3
\]

and hence \( 6m + 2k = v + 3 \), which has no solutions for \( m \) and \( k \) when \( v \equiv 0 \pmod{6} \).

\((\Leftarrow)\) (i) If \( v \equiv 1 \) or \( 3 \pmod{6} \) and \( v \neq 9 \), then the existence of cyclic \( STS(v) \)'s implies the existence of cyclic \( E(v;v) \)'s.

(ii) If \( v \equiv 3 \pmod{6} \) and \( v \neq 9 \), then a set of base blocks of a cyclic \( STS(v) \) based on \( \mathbb{Z}_v \), except for the base block \( \{0,v/3,2v/3\} \), together with a block \( \{0,0,v/3\} \), forms a collection of base blocks of a cyclic \( E(v;0) \). When \( v = 9 \), \( \{0,0,2\} \) and \( \{0,1,4\} \) form a collection of base blocks of a cyclic \( E(9;0) \).

3. Rotational Extended Triple Systems

Throughout this section, we will assume that our \( k \)-rotational \( E(v;n) \) is based on \( (\mathbb{Z}_{v-1}/k \times \mathbb{Z}_k) \cup \{\infty\} \) and the corresponding \( k \)-rotational automorphism is \( \alpha = (\infty)((0,0)(1,0)\cdots((v-1)/k-1,0))((0,1)(1,1)\cdots((v-1)/k-1,1))\cdots((0,k-1)(1,k-1)\cdots((v-1)/k-1,k-1)). \) For brevity, we will write \( x_i \) for the ordered pair \( (x,i) \) and \( \mathbb{Z}_{v-1} \cup \{\infty\} \) instead of \( (\mathbb{Z}_{v-1} \times \mathbb{Z}_1) \cup \{\infty\} \). By an elementary argument, it is easy to see the following lemma.

Lemma 3.1. If there exists a \( k \)-rotational \( E(v;n) \), then \( n = t((v-1)/k) + 1 \) for \( t = 0,1,\cdots \) or \( k \).

Lemma 3.2. A necessary condition for the existence of a \( 1 \)-rotational \( E(v;n) \) is

(i) \( n = v \) and \( v \equiv 3 \) or \( 9 \pmod{24} \) or

(ii) \( n = 1 \) and \( v \equiv 1 \) or \( 2 \pmod{3} \).

Proof. (i) It follows from the fact the class of \( 1 \)-rotational \( E(v;v) \)'s is the class of \( 1 \)-rotational \( STS(v) \)'s.

(ii) It follows from the necessary condition for the existence of an \( E(v;1) \).
The following theorem is a consequence of the spectrum for 1-rotational
STS(v)'s.

**Theorem 3.3.** A 1-rotational $E(v; v)$ exists if and only if $v \equiv 3$ or $9$
(mod 24).

**Lemma 3.4.** There is no 1-rotational $E(10; 1)$.

*Proof.* If there were a 1-rotational $E(10; 1)$, then it must contain base
blocks of the forms $\{\infty, \infty, \infty\}$ and $\{\infty, 0, 0\}$. Deleting these base blocks
would yield a cyclic $STS(9)$ which does not exist.

**Lemma 3.5.** If $v \equiv 4$ (mod 6) and $v \neq 10$, then there exists a 1-rotational
$E(v; 1)$.

*Proof.* If $v = 6t + 4$ and $t \neq 1$, then a collection of base blocks of a cyclic
$STS(6t + 3)$ based on $\mathbb{Z}_{6t+3}$, together with two triples $\{\infty, \infty, \infty\}$ and
$\{\infty, 0, 0\}$, forms a collection of base blocks of a 1-rotational $E(v; 1)$.

**Definition 3.6.** A $(H, k)$-system is a set of ordered pairs $\{(a_r, b_r) | r = \, 1, 2, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \ldots, k$ and
$\bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 2, \ldots, k + 1, k + 3, k + 4, \ldots, 2k + 1\}$.

**Lemma 3.7.** If $k \equiv 1$ or $2$ (mod 4), then there exists a $(H, k)$-system.

*Proof. Case 1.* $k = 4t + 1$.

\[
\begin{align*}
t & = 0 : (1, 2). \\
t & = 1 : (10, 11), (2, 4), (6, 9), (1, 5), (3, 8). \\
t & > 1 : \\
& \quad (r, 4t + 2 - r), \quad r = 1, 2, \ldots, 2t, \\
& \quad (4t + 2 + r, 8t + 4 - r), \quad r = 1, 2, \ldots, t - 1, \\
& \quad (5t + 2 + r, 7t + 3 - r), \quad r = 1, 2, \ldots, t - 1, \\
& \quad (2t + 1, 6t + 2), (4t + 2, 6t + 3), (7t + 3, 7t + 4). 
\end{align*}
\]

**Case 2.** $k = 4t + 2$

\[
\begin{align*}
t & = 0 : (1, 2), (3, 5) \\
t & = 1 : (11, 12), (3, 5), (10, 13), (2, 6), (4, 9), (1, 7). \\
t & > 1 : \\
& \quad (r, 4t + 4 - r), \quad r = 1, 2, \ldots, 2t + 1, \\
& \quad (4t + 4 + r, 8t + 5 - r), \quad r = 1, 2, \ldots, t - 1, \\
& \quad \vdots \\
& \quad (7t + 3, 7t + 4). 
\end{align*}
\]
Lemma 3.8. If \( v \equiv 13 \text{ or } 19 \pmod{24} \), then there exists a 1-rotational \( E(v;1) \).

Proof. Let \( v = 6t + 1 \) and \( t \equiv 2 \text{ or } 3 \pmod{4} \). Then the following triples form a collection of base blocks of a 1-rotational \( E(v;1) \):

\[
\begin{align*}
(5t + 3 + r, 7t + 4 - r), & \quad r = 1, 2, \ldots, t - 1, \\
(2t + 2, 6t + 3), (6t + 4, 8t + 5), (7t + 4, 7t + 5).
\end{align*}
\]

Lemma 3.9. An \((I, k)\)-system is a set of ordered pairs \( \{(a_r, b_r)\mid r = 1, 2, \ldots, k\} \) such that \( b_r - a_r = r \) for \( r = 1, 2, \ldots, k \) and \( \bigcup_{r=1}^{k} \{a_r, b_r\} = \{1, 2, \ldots, k+1, k+3, k+4, \ldots, (3k+1)/2 + 1, (3k+1)/2 + 3, (3k+1)/2 + 4, \ldots, 2k+2\} \).

Lemma 3.10. If \( k \) is an odd integer, then there exists an \((I, k)\)-system.

Proof. Case 1. \( k = 4t + 1 \).

\[
\begin{align*}
(r, 4t + 3 - r), & \quad r = 1, 2, \ldots, 2t + 1, \\
(4t + 3 + r, 8t + 5 - r), & \quad r = 1, 2, \ldots, 2t
\end{align*}
\]

Case 2. \( k = 4t + 3 \).

\[
\begin{align*}
(r, 4t + 5 - r), & \quad r = 1, 2, \ldots, 2t + 2, \\
(4t + 5 + r, 8t + 9 - r), & \quad r = 1, 2, \ldots, 2t + 1.
\end{align*}
\]

Lemma 3.11. If \( v \equiv 1 \pmod{24} \), then there exists a 1-rotational \( E(v;1) \).

Proof. If \( v = 6t + 1 \) and \( t \equiv 0 \pmod{4} \), \( t > 0 \), then the following triples form a collection of base blocks of a 1-rotational \( E(v;1) \):

\[
\begin{align*}
\{\infty, \infty, \infty\}, \{\infty, 0, 3t\}, \{0, 2t, 4t\}, \{0, 0, (5t)/2\}, \\
\{0, r, b_r + t - 1\} & \mid r = 1, 2, \ldots, t - 1
\end{align*}
\]
where $\{(a_r, b_r) | r = 1, 2, \cdots, t - 1\}$ is an $(I, t - 1)$-system. When $t = 0$, $\{\infty, \infty, \infty\}$ forms a 1-rotational $E(1; 1)$.

**Definition 3.12.** A $(J, k)$-system is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \cdots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \cdots, k$ and $\bigcup_{r=1}^{k}\{a_r, b_r\} = \{1, 2, \cdots, k/2, k/2 + 2, k/2 + 3, \cdots, k + 1, k + 3, k + 4, \cdots, 2k + 2\}$.

**Lemma 3.13.** If $k$ is an even integer, then there exists a $(J, k)$-system.

**Proof.** Case 1. $k = 4t$.

$$(r, 4t + 2 - r), \quad r = 1, 2, \cdots, 2t,$$

$$(4t + 2 + r, 8t + 3 - r), \quad r = 1, 2, \cdots, 2t$$

Case 2. $k = 4t + 2$.

$$(r, 4t + 4 - r), \quad r = 1, 2, \cdots, 2t + 1,$$

$$(4t + 4 + r, 8t + 7 - r), \quad r = 1, 2, \cdots, 2t + 1.$$  

**Lemma 3.14.** If $v \equiv 7 \pmod{24}$, then there exists a 1-rotational $E(v; 1)$.

**Proof.** If $v = 6t + 1$ and $t \equiv 1 \pmod{4}$, then the following triples form a collection of base blocks of a 1-rotational $E(v; 1)$:

$$\{\infty, \infty, \infty\}, \{\infty, 0, 3t\}, \{0, 2t, 4t\}, \{0, 0, 3(t - 1)/2 + 1\},$$

$$\{0, r, b_r + t - 1\} \mid r = 1, 2, \cdots, t - 1, \quad t > 1,$$

where $\{(a_r, b_r) | r = 1, 2, \cdots, t - 1\}$ is a $(J, t - 1)$-system.

**Lemma 3.15.** If $v \equiv 2 \pmod{6}$, then there exists a 1-rotational $E(v; 1)$.

**Proof.** If $v = 6t + 2$, then a collection of base blocks of a cyclic $STS(6t + 1)$ based on $Z_{6t+1}$, together with two triples $\{\infty, \infty, \infty\}$ and $\{\infty, 0, 0\}$, forms a collection of a base blocks of a 1-rotational $E(v; 1)$.

An $(A, k)$-system (a $(B, k)$-system, respectively) [7] is a set of ordered pairs $\{(a_r, b_r) | r = 1, 2, \cdots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \cdots, k$ and $\bigcup_{r=1}^{k}\{a_r, b_r\} = \{1, 2, \cdots, 2k\} (= \{1, 2, \cdots, 2k - 1, 2k + 1\}$, respectively). An $(A, k)$-system a $(B, k)$-system are essentially the same as a Skolem $k$-sequence and a hooked Skolem $k$-sequence, respectively [4,8]. It is well-known that an $(A, k)$-system exists if and only if $k \equiv 0$ or $1 \pmod{4}$, and a $(B, k)$-system exists if and only $k \equiv 2$ or $3 \pmod{4}$ [see 4,7,8].
Lemma 3.16. If \( v \equiv 5 \pmod{6} \), then there exists a 1-rotational \( E(v;1) \).

Proof. If \( v = 6t + 5 \), then the following triples form a collection of base blocks of a 1-rotational \( E(v;1) \):

\[
\{\infty, \infty, \infty\}, \{\infty, 0, 3t + 2\}, \{0, 0, 3t + 1\} \quad \text{if } t \equiv 0 \text{ or } 1 \pmod{4},
\{\infty, \infty, \infty\}, \{\infty, 0, 3t + 2\}, \{0, 0, 3t\} \quad \text{if } t \equiv 2 \text{ or } 3 \pmod{4},
\{0, r, b_r + t\} | r = 1, 2, \ldots, t
\]

where \( \{(a_r, b_r) | r = 1, 2, \ldots, t\} \) is an \((A,t)\)-system of a \((B,t)\)-system depending on whether \( t \equiv 0, 1 \pmod{4} \) or \( t \equiv 2, 3 \pmod{4} \).

We now can conclude the following theorem.

Theorem 3.17. A 1-rotational \( E(v;1) \) exists if and only if \( v \equiv 1 \) or \( 2 \pmod{3} \) and \( v \neq 10 \).

Now, let us construct 2-rotational \( E(v;n) \)'s. First of all it follows from Lemma 3.1 that if there exists a 2-rotational \( E(v;n) \) then \( n = 1, (v+1)/2 \) or \( v \).

Lemma 3.18. A necessary condition for the existence of a 2-rotational \( E(v;n) \) is

(i) \( n = v \) and \( v \equiv 1, 3, 7, 9, 15 \text{ or } 19 \pmod{24} \) or
(ii) \( n = (v+1)/2 \) and \( v \equiv 1 \pmod{6} \) or
(iii) \( n = 1 \) and \( v \equiv 1 \) or \( 5 \pmod{6} \).

Proof. (i) It follows from the fact that the class of 2-rotational \( E(v;v) \)'s is the class of 2-rotational \( STS(v) \)'s.

(ii) Since the existence of an \( E(v;n) \) implies \( n \equiv 0 \text{ of } 1 \pmod{3}, (v+1)/2 \equiv 0 \text{ or } 1 \pmod{3} \). If \( (v+1)/2 \equiv 0 \pmod{3} \), then \( v \equiv 5 \pmod{6} \). But when \( n \equiv 0 \pmod{3} \), we must have \( v \equiv 0 \pmod{3} \). Thus, \( (v+1)/2 \equiv 0 \pmod{3} \) is impossible. We then have \( (v+1)/2 \equiv 1 \pmod{3} \) and hence \( v \equiv 1 \pmod{6} \).

(iii) If \( n = 1 \), then \( v \equiv 1 \) or \( 2 \pmod{3} \). But, since \( (v-1)/2 \) is an integer, \( v \equiv 1 \) or \( 5 \pmod{6} \).

The following theorem is a consequence of the spectrum for 2-rotational \( STS(v) \)'s.

Theorem 3.19. A 2-rotational \( E(v;v) \) exists if and only if \( v \equiv 1, 3, 7, 9, 15 \text{ or } 19 \pmod{24} \).

If \( v \equiv 1 \) or \( 5 \pmod{6} \), then \( v-1 \equiv 0 \pmod{2} \). Thus, for any permutation \( \alpha \) of type \([\alpha] = [1, 0, \cdots, 0, 1, 0] \), \( \alpha^2 \) is of type \([1, 0, \cdots, 0, 2, 0, \cdots, 0] \),
i.e. $\alpha_{(v-1)/2} = 2$. Hence the following theorem follows from the existence of 1-rotational $E(v; 1)$'s and Lemma 3.18.

**Theorem 3.20.** A 2-rotational $E(v; 1)$ exists if and only if $v \equiv 1$ or $5 \pmod{6}$.

**Lemma 3.21.** There exists a 2-rotational $E(19; 10)$.

**Proof.** The following triples form a collection of base blocks of a 2-rotational $E(19; 10)$:

$$\{\infty, \infty, \infty\}, \{0_0, 0_0, 0_0\}, \{\infty, 0_0, 0_1\}, \{0_1, 0_1, 4_1\},$$
$$\{0_1, 2_1, 8_1\}, \{0_0, 1_0, 2_1\}, \{0_0, 2_0, 7_1\}, \{0_0, 3_0, 6_1\},$$
$$\{0_0, 4_0, 8_1\}.$$

**Lemma 3.22.** If $v \equiv 7 \pmod{12}$, then there exists a 2-rotational $E(v; (v+1)/2)$.

**Proof.** Let $v = 12t + 7$. If $t = 1$, then it has been treated in Lemma 3.21. If $t > 1$, then the following triples form a collection of base blocks of a 2-rotational $E(v; (v+1)/2)$:

$$\{\infty, \infty, \infty\}, \{0_0, 0_0, 0_0\}, \{\infty, 0_0, 0_1\}, \{0_1, 0_1, (2t + 1)_1\}$$
if $t \equiv 0$ or $1 \pmod{4}$,

$$\{\infty, \infty, \infty\}, \{0_0, 0_0, 0_0\}, \{\infty, 0_0, (6t + 2)_1\}, \{0_1, 0_1, (2t + 1)_1\}$$
if $t \equiv 2$ or $3 \pmod{4}$,

$$\{0_0, r_0, (b_r)_1\}|r = 1, 2, \cdots, 3t + 1\}$$

where $\{a_r, b_r\}|r = 1, 2, \cdots, 3t + 1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod{4}$ or $t \equiv 2, 3 \pmod{4}$;

A collection of all base blocks of a cyclic $STS(6t+3)$ based on $Z_{6t+3} \times \{1\}$, except the base block $\{0_1, (2t+1)_1, (4t+2)_1\}$. 

**Definition 3.23.** A $(K, k)$-system is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \cdots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \cdots, k$ and $\bigcup_{r=1}^{k}\{a_r, b_r\} = \{1, 2, \cdots, k - 1, k + 1, k + 2, \cdots, 2k + 1\}$.

**Lemma 3.24.** If $k \equiv 1$ or $2 \pmod{4}$, then there exists a $(K, k)$-system.

**Proof.** from a $(K, k)$-system.
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**Proof. Case 1.** $k = 4t + 1.$

\[
t = 0 : (2, 3).
\]
\[
t > 0 : \\
(r, 4t + 1 - r), \quad r = 1, 2, \ldots, t - 1(t > 1), \\
(t + 1 + r, 3t + 2 - r), \quad r = 1, 2, \ldots, t - 1(t > 1), \\
(4t + 2 + r, 8t + 4 - r), \quad r = 1, 2, \ldots, 2t, \\
(2t + 1, 4t + 2), (2t + 2, 6t + 3), (t, t + 1).
\]

**Case 2.** $k = 4t + 2.$

\[
t = 0 : (4, 5), (1, 3). \\
t > 0 : \\
(r, 4t + 2 - r), \quad r = 1, 2, \ldots, 2t, \\
(4t + 3 + r, 8t + 6 - r), \quad r = 1, 2, \ldots, t, \\
(5t + 3 + r, 7t + 4 - r), \quad r = 1, 2, \ldots, t - 1(t > 1), \\
(2t + 1, 6t + 3), (4t + 3, 6t + 4), (7t + 4, 7t + 5).
\]

**Lemma 3.25.** If $v \equiv 13$ or $25 \pmod{48},$ then there exists a 2-rotational $E(v; (v + 1)/2).$

**Proof.** If $v = 12t + 1$ and $t \equiv 1$ or $2 \pmod{4},$ then the following triples from a collection of base blocks of a 2-rotational $E(v; (v + 1)/2):$

\[
\{\infty, \infty, \infty\}, \{\infty, 0_0, (3t)_0\}, \{\infty, 0_1, (3t)_1\}, \{0_0, 0_0, 0_0\}, \\
\{0_0, 0_1, (3t - 1)_1\}, \{0_1, 0_1, (3t - 2)_1\}, \\
\{(0_0, r_0, (b_r)_1) | r = 1, 2, \ldots, 3t - 1\}
\]

where $\{(a_r, b_r)|r = 1, 2, \ldots, 3t - 1\}$ is a $(K, 3t - 1)$-system,

\[
\{(0_1, r_1, (b_r + t - 1)_1) | r = 1, 2, \ldots, t - 1\}
\]

where $\{(a_r, b_r)|r = 1, 2, \ldots, t - 1\}$ is an $(A, t - 1)$-system.

**Definition 3.26.** A $(G, k)$-system is a set of ordered pairs $\{(a_r, b_r)|r = 1, 2, \ldots, k\}$ such that $b_r - a_r = r$ for $r = 1, 2, \ldots, k$ and $\cup_{r=1}^{k}\{a_r, b_r\} = \{1, 2, \ldots, k/2, k/2 + 2, k/2 + 3, \ldots, 2k + 1\}.$

**Lemma 3.27.** If $k \equiv 0 \pmod{2},$ then there exists a $(G, k)$-system.
Proof. Case 1. \( k = 4t \).

\[(4t + 1 + r, 8t + 2 - r), \quad r = 1, 2, \ldots, 2t\]
\[(r, 4t + 2 - r), \quad r = 1, 2, \ldots, 2t.\]

Case 2. \( k = 4t + 2 \).

\[(4t + 3 + r, 8t + 6 - r), \quad r = 1, 2, \ldots, 2t + 1,\]
\[(r, 4t + 4 - r), \quad r = 1, 2, \ldots, 2t + 1.\]

**Definition 3.28.** A \((L, k)\)-system is a set of ordered pairs \(\{(a_r, b_r)|r = 1, 2, \ldots, k\}\) such that \(b_r - a_r = r\) for \(r = 1, 2, \ldots, k\) and \(\bigcup_{r=1}^{k}\{a_r, b_r\} = \{1, 2, \ldots, k/2 + 1, k/2 + 3, k/2 + 4, \ldots, k + 2, k + 4, k + 5, \ldots, 2k + 2\}\).

**Lemma 3.29.** If \(k \equiv 0 \pmod{2}\), then there exists a \((L, k)\)-system.

**Proof.** If \(k \equiv 0 \pmod{2}\), then the following ordered pairs form a \((L, k)\)-system:

\[(2 + r, k + 3 - r), \quad r = 1, 2, \ldots, k/2 - 1,\]
\[(k + 3 + r, 2k + 3 - r), \quad r = 1, 2, \ldots, k/2 - 1,\]
\[(1, 2), (k/2 + 3, (3k)/2 + 3).\]

**Lemma 3.30.** If \(v \equiv 37 \pmod{48}\), then there exists a 2-rotational \(E(v; (v + 1)/2)\).

**Proof.** If \(v = 12t + 1\) and \(t \equiv 3 \pmod{4}\), then the following triples form a collection of base blocks of a 2-rotational \(E(v; (v + 1)/2)\):

\[\{\infty, \infty, \infty\}, \{0_0, 0_0, 0_0\}, \{\infty, 0_0, (3t)_0\}, \{\infty, 0_1, (3t)_1\},\]
\[\{0_0, 0_1, ((3t + 1)/2)_1\}, \{0_1, 0_1, (2t + 1)_1\},\]
\[\{(0_0, r_0, (b_r)_1)|r = 1, 2, \ldots, 3t - 1\}\]

where \(\{(a_r, b_r)|r = 1, 2, \ldots, 3t - 1\}\) is a \((G, 3t - 1)\)-system,

\[\{0_1, r_1, (b_r + t - 1)_1|r = 1, 2, \ldots, t - 1\}\]

where \(\{(a_r, b_r)|r = 1, 2, \ldots, t - 1\}\) is a \((L, t - 1)\)-system.
Definition 3.31. A \((M, k)\)-system is a set of ordered pairs \(\{(a_r, b_r)|r = 1, 2, \ldots, k\}\) such that \(b_r - a_r = r\) for \(r = 1, 2, \ldots, k\) and \(\bigcup_{r=1}^k\{a_r, b_r\} = \{1, 2, \ldots, (k + 1)/2, (k + 1)/2 + 2, (k + 1)/2 + 3, \ldots, k + 1, k + 3, k + 4, \ldots, 2k + 2\}\).

Lemma 3.32. If \(k \equiv 1 \pmod{2}\) and \(k \neq 1\), then there exists a \((M, k)\)-system.

Proof. If \(k \neq 1\) is an odd integer, then the following ordered pairs form a \((M, k)\)-system:

\[
(2 + r, k + 2 - r), \quad r = 1, 2, \ldots, (k - 1)/2 - 1,
(2k + 3 - r, 3k + 1)/2 + 2, (k + 1)/2 + 3, \ldots, k + 1, k + 3, k + 4, \ldots, 2k + 2)\).
\]

Definition 3.33 An \((E, k)\)-system is a set of ordered pairs \(\{(a_r, b_r)|r = 1, 2, \ldots, k\}\) such that \(b_r - a_r = r\) for \(r = 1, 2, \ldots, k\) and \(\bigcup_{r=1}^k\{a_r, b_r\} = \{1, 2, \ldots, (k + 1)/2 - 1, (k + 1)/2 + 1, (k + 1)/2 + 2, \ldots, 2k + 1\}\).

Lemma 3.34. If \(k \equiv 1 \pmod{2}\), then there exists an \((E, k)\)-system.

Proof. Case 1. \(k = 4t + 1\).

\[
(4t + 1 + r, 8t + 4 - r), \quad r = 1, 2, \ldots, 2t + 1,
(r, 4t + 2 - r), \quad r = 1, 2, \ldots, 2t.
\]

Case 2. \(k = 4t - 1\).

\[
(4t - 1 + r, 8t - r), \quad r = 1, 2, \ldots, 2t,
(r, 4t - r), \quad r = 1, 2, \ldots, 2t - 1.
\]

Lemma 3.35. If \(v \equiv 1 \pmod{48}\), then there exists a 2-rotational \(E(v; (v + 1)/2)\).

Proof. If \(v = 12t + 1, t \equiv 0 \pmod{4}\) and \(t \neq 0\), then the following triples form a collection of base blocks of a 2-rotational \(E(v; (v + 1)/2)\):

\[
\{(\infty, 0, 0), (0, 0, 0), (\infty, 0, (3t)0), (\infty, 0, (3t)1),
\{0, 0 (3t)2}, \{0, 0, (3t)1), \{0, 0, (2t)1\}
\]
where \( \{(a_r, b_r) | r = 1, 2, \ldots, 3t - 1\} \) is a \((E, 3t - 1)\)-system,

\[ \{(0_1, r_1, (b_r + t - 1)_1) | r = 1, 2, \ldots, t - 1\} \]

where \( \{(a_r, b_r) | r = 1, 2, \ldots, t - 1\} \) is a \((M, t - 1)\)-system.

We can now obtain the following theorem.

**Theorem 3.36.** A 2-rotational \( E(v; (v + 1)/2) \) exists if and only if \( v \equiv 1 \pmod{6} \).

In the rest of this paper, we will construct some 3-rotational \( E(v; n) \)'s. From Lemma 3.1, if there exists a 3-rotational \( E(v; n) \) then \( n = 1, (v + 2)/3, (2v + 1)/3 \) or \( v \).

**Lemma 3.37.** A necessary condition for the existence of a 3-rotational \( E(v; n) \) is

(i) \( n = v \) and \( v \equiv 1 \) or 19 \((\pmod{24})\) or

(ii) \( n = 1 \) and \( v \equiv 1 \) \((\pmod{3})\) or

(iii) \( n = (v + 2)/3 \) and \( v \equiv 1 \) \((\pmod{9})\) or

(iv) \( n = (2v + 1)/3 \) and \( v \equiv 1 \) \((\pmod{18})\).

**Proof.** (i) It follows form the fact that the class of 3-rotational \( E(v; v) \)'s is the class of 3-rotational \( STS(v) \)'s.

First of all, \( v \equiv 1 \) \((\pmod{3})\) since the existence of a 3-rotational \( E(v; n) \) implies \( (v - 1)/3 \) to be an integer.

(ii) It is obvious.

(iii) Since \( v \equiv 1 \) \((\pmod{3})\), \( n \equiv 1 \) \((\pmod{3})\) and hence \( (v + 2)/3 \equiv 1 \) \((\pmod{3})\). So \( v \equiv 1 \) \((\pmod{9})\).

(iv) In this case, \( (2v + 1)/3 \equiv 1 \) \((\pmod{3})\) and hence \( v \equiv 1 \) \((\pmod{9})\). If \( v \equiv 10 \) \((\pmod{18})\), then \( v \) is even; so we must have \( n \leq v/2 \). But \( (2v + 1)/3 > v/2 \). Thus, \( v \equiv 10 \) \((\pmod{18})\) is impossible.

The following theorem is a consequence of the spectrum for 3-rotational \( STS(v) \)'s.

**Theorem 3.38.** A 3-rotational \( E(v; v) \) exists if and only if \( v \equiv 1 \) or 19 \((\pmod{24})\).

**Lemma 3.39.** There exists a 3-rotational \( E(10; 1) \).
Proof. The following triples form a collection of base blocks of a 3-rotational:

\[
\{\infty, \infty, \infty\}, \{\infty, 0_2, 0_2\}, \{\infty, 0_0, 0_1\}, \{0_1, 0_1, 0_2\},
\{0_0, 0_0, 0_2\}, \{0_0, 1_0, 2_1\}, \{0_1, 1_1, 2_1\}, \{0_2, 1_2, 2_0\}.
\]

If \(v \equiv 1 \pmod{3}\), then \(v - 1 \equiv 0 \pmod{3}\). Thus, for any permutation \(\alpha\) of type \([\alpha] = [1, 0, \ldots, 1, 0]\), \(\alpha^3\) is of type \([1, 0, \ldots, 0, 3, 0, 0, \ldots, 0]\), i.e. \(\alpha_{(v-1)/3} = 3\). Hence the following theorem follows from the existence of 1-rotational \(E(v; 1)\)'s and Lemma 3.39.

**Theorem 3.40.** A 3-rotational \(E(v; 1)\) exists if and only if \(v \equiv 1 \pmod{3}\).

**Definition 3.41.** A \((N, 3t-1)\)-system is a set of ordered pairs \(\{(a_r, b_r) | r = 1, 2, \ldots, 2t-1, 2t + 2, \ldots, 3t - 1\}\) such that \(b_r - a_r = r\) for \(r = 1, 2, \ldots, 2t-1, 2t+1, 2t+2, \ldots, 3t-1\) and \(\bigcup_{r=1, r \neq 2t}^{3t-1} \{a_r, b_r\} = \{-t/2, -t/2 + 1, \ldots, -1, 1, 2, \ldots, 3t-1, 3t+1, 3t+2, \ldots, 4t-1, 4t+1, 4t+2, \ldots, 5t-1, 5t+1, 5t+2, \ldots, 6t - t/2 - 1\}\).

**Lemma 3.42.** If \(t \equiv 0 \pmod{4}\) and \(t \neq 0\), then there exists a \((N, 3t-1)\)-system.

**Proof.** Let \(t \equiv 0 \pmod{4}\) and \(t \neq 0\).

\[
t = 4: (5,6), (15,17), (1,4), (14,18), (2,7), (13,19), (3,10),
\]
\[
(-1,8), (11,21), (-2,9).
\]

\[
t > 4:
\]
\[
(4t - r, 4t + r), \quad r = 1, 2, \ldots, t-1, t+1, t+2, \ldots (3t-2)/2,
\]
\[
(-(t+2)/2 + r, 5t/2 - r), r = 1, 2, \ldots, t/2,
\]
\[
(1 + r, 2t - r), \quad r = 1, 2, \ldots, (t-2)/2,
\]
\[
((t+4)/2 + r, (3t + 2)/2 - r), r = 1, 2, \ldots, (t-4)/4,
\]
\[
(t - r, t + 1 + r), \quad r = 1, 2, \ldots, (t-8)/4, (t > 8),
\]
\[
(1, t), ((t+2)/2, 5t/2), ((t+4)/2, t+1), (5t/4, (5t + 4)/4).
\]

**Lemma 3.43.** If \(v \equiv 1 \pmod{72}\), then there exists a 3-rotational \(E(v; (v+2)/3)\).
Proof. If \( v = 18t + 1 \) and \( t \equiv 0 \pmod{4} \), then the following triples form a collection of base blocks of a 3-rotational \( E(v; (v+2)/3) \):

(i) \( \{0_0, 0_0, (2t)_0\} \),
(ii) \( \{\infty, \infty, \infty\}, \{0_1, 0_1, 0_1\}, \{0_1, (2t)_1, (4t)_1\}, \{0_2, 0_2, (2t)_2\}, \{\infty, 0_0, (3t)_0\}, \{\infty, 0_1, (3t)_1\}, \{\infty, 0_2, (3t)_2\}, \{0_0, 0_1, 0_2\}, \{(3t)_1, t_2, 0_0\}, \{(4t)_1, (3t)_2, 0_0\}, \{(5t)_1, (2t)_2, 0_0\} \).
(iii) \( \{(0_i, r_i, (b_r)_{i+1})\} | i \in \mathbb{Z}_3, r = 1, 2, \cdots, 2t - 1, 2t + 1, 2t + 2, \cdots, 3t - 1 \)
where \( \{(a_r, b_r) | r = 1, 2, \cdots, 2t - 1, 2t + 1, 2t + 2, \cdots, 3t - 1 \} \) is a \((N, 3t - 1)\)-system.

Lemma 3.44. If \( v \equiv 1 \pmod{72} \), then there exists a 3-rotational \( E(v; (2v+1)/3) \).

Proof. The base blocks in Lemma 3.43, except the base block \( \{0_0, 0_0, (2t)_0\} \) in (i), together with two triples \( \{0_0, 0_0, 0_0\} \) and \( \{0_0, (2t)_0, (4t)_0\} \), form a collection of base blocks of a 3-rotational \( E(v; (2t + 1)/3) \).

Definition 3.45. An \((0, 3t - 1)\)-system is a set of ordered pairs \( \{(a_r, b_r) | r = 1, 2, \cdots, 3t - 1 \} \) such that \( b_r - a_r = r \) for \( r = 1, 2, \cdots, 3t - 1 \) and \( \cup_{r=1}^{3t-1} \{(a_r, b_r) | r = 1, 2, \cdots, 3t - 1 \} \) such that \( b_r - a_r = r \) for \( r = 1, 2, \cdots, 3t - 1 \) and \( \cup_{r=1}^{3t-1} \{(a_r, b_r) | r = 1, 2, \cdots, 3t - 1 \} \) is a \((N, 3t - 1)\)-system.

Lemma 3.46. If \( t \equiv 1 \pmod{4} \), then there exists an \((0, 3t - 1)\)-system.

Proof. Let \( t \equiv 1 \pmod{4} \).

\[
\begin{align*}
t &= 1 : (1, 2) . \\
t &= 5 : \\
&\quad (20 - r, 20 + r), \quad r = 1, 2, 3, 4, 6, 7; \\
&\quad (4, 5), (6, 9), (2, 7), (1, 8), (3, 12), (-1, 10), (-2, 11) . \\
t &> 5 : \\
&\quad (4t - r, 4t + r), \quad r = 1, 2, \cdots, 3t - 1, t = 1, t + 2, \cdots, (3t - 1)/2, \\
&\quad (t + 1)/2 + r, (5t - 1)/2 - r, 1, 2, \cdots, (t - 1)/2, \\
&\quad (1 + r, 2t - r), \quad r = 1, 2, \cdots, t - 3/2, \\
&\quad ((t + 3)/2 + r, (3t + 3)/2 - r), 1, 2, \cdots, (t - 1)/4, \\
&\quad (t - r, t + 1 + r), \quad r = 1, 2, \cdots, (t - 9)/4, (t > 9), \\
&\quad (1, t + 1), ((t + 1)/2, (5t - 1)/2), (5t - 1)/4, (5t + 3)/4).
\end{align*}
\]
Lemma 3.47. If \( v \equiv 19 \pmod{72} \), then there exists a 3-rotational \( E(v; (v+2)/3) \).

Proof. If we replace \( t \equiv 0 \pmod{4} \) and \((N,3t-1)\)-system in Lemma 3.43 by \( t \equiv 1 \pmod{4} \) and \((0,3t-1)\)-system, respectively, then we have a collection of base blocks of a 3-rotational \( E(v; (v+2)/3) \) where \( v \equiv 19 \pmod{72} \).

Lemma 3.48. If \( v \equiv 19 \pmod{72} \), then there exists a 3-rotational \( E(v; (2v+1)/3) \).

Proof. If we replace \( t \equiv 0 \pmod{4} \) and \((N,3t-1)\)-system in Lemma 3.44 by \( t \equiv 1 \pmod{4} \) and \((0,3t-1)\)-system, respectively, then we have a collection of base blocks of a 3-rotational \( E(v; (2v+1)/3) \) where \( v \equiv 19 \pmod{72} \).

Lemma 3.49. If \( v \equiv 10 \pmod{18} \), then there is no 3-rotational \( E(v; (v+2)/3) \).

Proof. If \( v \equiv 10 \pmod{18} \), then \((v-1)/3\) is odd. Thus, if there were a 3-rotational \( E(v; (v+2)/3) \) then it would contain a base block of the form \( \{\infty, a_i, b_j\} \) for some \( i, j \in \mathbb{Z}_3 \) with \( i \neq j \). Hence there were no blocks containing the type of pairs \( \{\infty, x_t\} \) where \( t \in \mathbb{Z}_3 \) and \( t \neq i, j \).

In the existence problem for 3-rotational \( E(v;n) \)'s, the following case remains open:
If \( v \equiv 37 \) or \( 55 \pmod{72} \), does there exist a 3-rotational \( E(v;n) \), where \( n = (v+2)/3 \) or \( (2v+1)/3 \)?

References


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1. Introduction

For a given order, the concept of a topological space is closely related to the corresponding class of sets.

Let $X$ be a topological space.

1. $X$ is compact.

2. Every open set is contained in a compact set.

3. Every closed and bounded set is compact.

4. Every compact subset of $X$ is closed.

5. Every open cover of $X$ has a finite subcover.

6. Every family of open sets with the finite intersection property has a non-empty intersection.

7. Every locally compact space is a Baire space.

8. Every locally compact Hausdorff space is completely regular.

9. Every metric space is a Hausdorff space.

10. Every compact metric space is complete.

11. Every complete metric space is separable.

12. Every compact Hausdorff space is normal.

13. Every normal Hausdorff space is completely regular.

14. Every completely regular Hausdorff space is normal.

15. Every normal space is regular.

16. Every regular space is separable.

17. Every separable space is normal.

18. Every normal space is normalizable.

19. Every normalizable space is normal.

20. Every normalizable space is metrizable.

21. Every metrizable space is normalizable.

22. Every normalizable space is completely regular.

23. Every completely regular space is normalizable.

24. Every normalizable space is normal.

25. Every normal space is regular.

26. Every regular space is separable.

27. Every separable space is regular.

28. Every regular space is normalizable.

29. Every normalizable space is normal.

30. Every normalizable space is metrizable.

31. Every metrizable space is normalizable.

32. Every normalizable space is completely regular.

33. Every completely regular space is normalizable.

34. Every normalizable space is normal.

35. Every normalizable space is normalizable.

36. Every normalizable space is normalizable.

37. Every normalizable space is normalizable.

38. Every normalizable space is normalizable.

39. Every normalizable space is normalizable.

40. Every normalizable space is normalizable.