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A NOTE ON F-CLOSED SPACES

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1. Introduction

In 1969, Porter and Thomas [12] defined a topological space X to be quasi H-closed if every open cover of X has a finite proximate subcover. A family of sets whose union is dense in X is called a proximate cover of X. Recently, Chae and Lee [2] have introduced and studied the concept of F-closed spaces by utilizing feebly open sets due to Maheshwari and Tapi [6]. A topological space X is said to be F-closed [2] if every feebly open cover of X has a finite proximate subcover. The main purpose of the present note is to show that the F-closed property is equivalent to the quasi H-closed property.

2. Preliminaries

Throughout the present note, spaces always mean topological spaces on which no separation axioms are assumed. Let A be a subset of a space X. The closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be *preopen* [9] (resp. *semi-open* [5], *α-open* [10]) if $A \subset Int(Cl(A))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(Int(A)))$). The complement of a preopen (resp. semi-open, α-open) set is said to be *preclosed* (resp. *semi-closed, α-closed*). The intersection of all semi-closed sets of X containing A is called the *semi-closure* of A [3] and is denoted by $sCl(A)$. A subset A is said to be *feebly open* [6] if there exists an open set U of X such that $U \subset A \subset sCl(U)$. The following property is shown in [11, Lemma 3.1].

Lemma 2.1. *A subset of a space X is α-open in X if and only if it is semi-open and preopen in X.*

3. Quasi H-closed spaces

Lemma 3.1. *If A is a preopen set of a space X, then $Cl(Int(Cl(A))) = Cl(A)$ and $sCl(A) = Int(Cl(A))$.*

Proof. The first part is obvious and the second follows from $sCl(A) = A \cup Int(Cl(A))$ [1, Theorem 1.5].

Lemma 3.2. *A subset A of a space X is feebly open in X if and only if it is α-open in X.*

Proof. Let A be feebly open in X. There exists an open set U of X such that $U \subset A \subset sCl(U)$. Since $U \subset Int(A)$ and $sCl(U) = Int(Cl(U))$ by Lemma 3.1, we have $A \subset Int(Cl(Int(A)))$ and hence A is α-open in X. Conversely, let A be α-open in X. We have $A \subset Int(Cl(Int(A)))$ and hence $Int(A) \subset A \subset sCl(Int(A))$. Therefore, A is feebly open in X.

Definition 3.3. A space X is said to be *F-closed* [2] if every feebly open cover of X has a finite proximate subcover.

Theorem 3.4. *The following are equivalent for a space X:*

(a) *X is F-closed.*

(b) *X is quasi H-closed.*

(c) *Every preopen cover of X has a finite proximate subcover.*

(d) *For each family $\{F_\alpha | \alpha \in \nabla\}$ of preclosed sets in X satisfying $\cap\{F_\alpha | \alpha \in \nabla\} = \emptyset$, there exists a finite subset ∇_0 of ∇ such that $\cap\{Int(F_\alpha) | \alpha \in \nabla_0\} = \emptyset$.*

Proof. (a) $\Rightarrow$ (b) : The proof is obvious since every open set is feebly open.

(b) $\Rightarrow$ (c) : Let $\{V_\alpha | \alpha \in \nabla\}$ be a cover of X by preopen sets of X. For each $\alpha \in \nabla$, $V_\alpha \subset Int(Cl(V_\alpha))$ and $\{Int(Cl(V_\alpha)) | \alpha \in \nabla\}$ is an open cover of X. There exists a finite subset ∇_0 of ∇ such that

$$X = \cup\{Cl(Int(Cl(V_\alpha))) | \alpha \in \nabla_0\}.$$

By Lemma 3.1, we obtain $X = \cup\{Cl(V_\alpha) | \alpha \in \nabla_0\}$.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) : These follow easily from Lemmas 2.1 and 3.2.

Remark 3.5. By Theorem 3.4, we observe that Example 4.1 of [2] is false since $\beta\mathbf{N} \times \beta\mathbf{N}$ is compact.

A subset A of a space X is said to be *F-closed relative to X* [2] (resp. *quasi H-closed relative to X* [12]) if for every cover $\{V_\alpha | \alpha \in \nabla\}$ of A by feebly open (resp. open) sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{Cl(V_\alpha) | \alpha \in \nabla_0\}$.

Theorem 3.6. *A subset A of a space X is F-closed relative to X if and only if it is quasi H-closed relative to X.*

Proof. Suppose that A is quasi H-closed relative to X. Let $\{V_\alpha | \alpha \in \nabla\}$ be a cover of A by feebly open sets of X. By Lemma 3.2, $\{Int(Cl(Int(V_\alpha)))|\alpha \in \nabla\}$ is a cover of A by open sets of X. There exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{Cl(Int(V_\alpha))|\alpha \in \nabla_0\}$. By Lemmas 2.1 and 3.2, V_α is semi-open and hence $Cl(Int(V_\alpha)) = Cl(V_\alpha)$ for each $\alpha \in \nabla$. Therefore, we have $A \subset \cup\{Cl(V_\alpha)|\alpha \in \nabla_0\}$. The converse is obvious since every open set is feebly open.

It was pointed out in [12, p. 161] that every quasi H-closed subspace is quasi H-closed relative to the space but not conversely. In [2, Theorem 2.2], Chae and Lee showed that a feebly open subspace of a space X is F-closed if and only if it is F-closed relative to X. The following theorem is a slight improvement of this result since every feebly open set is preopen.

Theorem 3.7. *Let A be a preopen set of a space X. The subspace A is quasi H-closed if and only if A is quasi H-closed relative to X.*

Proof. Suppose that A is preopen in X and quasi H-closed relative to X. Let $\{V_\alpha | \alpha \in \nabla\}$ be a cover of A by open sets of the subspace A. For each $\alpha \in \nabla$, there exists an open set W_α of X such that $V_\alpha = W_\alpha \cap A$. Since A is preopen in X, we have

$$V_\alpha \subset W_\alpha \cap Int(Cl(A)) = Int(W_\alpha \cap Cl(A)) \subset Int(Cl(W_\alpha \cap A)) = Int(Cl(V_\alpha)).$$

Therefore, V_α is preopen in X and $\{Int(Cl(V_\alpha))|\alpha \in \nabla\}$ is a cover of A by open sets of X. By Lemma 3.1, there exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{Cl(V_\alpha)|\alpha \in \nabla_0\}$. Therefore, we obtain

$$A = \cup\{Cl(V_\alpha) \cap A|\alpha \in \nabla_0\} = \cup\{Cl_A(V_\alpha)|\alpha \in \nabla_0\},$$

where $Cl_A(V_\alpha)$ denotes the closure of V_α in the subspace A. This shows that A is quasi H-closed.

A function $f : X \to Y$ is said to be *α-continuous* [8] (resp. *α-irresolute* [7]) if $f^{-1}(V)$ is α-open in X for every open (resp. α-open) set V of Y. By Lemma 3.2, α-continuity (resp. α-irresoluteness) is equivalent to feeble continuity (resp. feeble irresoluteness) due to Chae and Lee [2]. A function $f : X \to Y$ is said to be *θ-continuous* [4] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(Cl(U)) \subset Cl(V)$.

Remark 3.8. For the properties on a function $f : X \to Y$, the following implications are known in [7] and [8]:

$$\begin{array}{lcl} \alpha - \text{irresoluteness} & & \\ & \searrow & \\ & & \alpha - \text{continuity} \;\Rightarrow\; \theta - \text{continuity}. \\ & \nearrow & \\ \text{continuity} & & \end{array}$$

Lemma 3.9. *If $f : X \to Y$ is θ-continuous and A is quasi H-closed relative to X, then $f(A)$ is quasi H-closed relative to Y.*

Proof. The proof is obvious and is thus omitted.

Corollary 3.10 (Chae and Lee [2]). *Let X be an F-closed space and $f : X \to Y$ a function. Then, the following properties hold:*

(a) *If f is a feebly continuous surjection, then Y is quasi H-closed.*

(b) *If f is a feebly irresolute surjection, then Y is F-closed.*

(c) *If f is feebly irresolute and Y is Hausdorff, then $f(X)$ is closed in Y.*

Proof. (a) and (b) are immediate consequences of Theorem 3.4 and Lemma 3.9. (c) follows from Theorem 3.4, Lemma 3.9 and the fact that if B is quasi H-closed relative to Y and Y is Hausdorff then B is closed in Y.

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