A NOTE ON \( F \)-CLOSED SPACES

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1. Introduction

In 1969, Porter and Thomas [12] defined a topological space \( X \) to be quasi \( H \)-closed if every open cover of \( X \) has a finite proximate subcover. A family of sets whose union is dense in \( X \) is called a proximate cover of \( X \). Recently, Chae and Lee [2] have introduced and studied the concept of \( F \)-closed spaces by utilizing feebly open sets due to Maheshwari and Tapi [6]. A topological space \( X \) is said to be \( F \)-closed [2] if every feebly open cover of \( X \) has a finite proximate subcover. The main purpose of the present note is to show that the \( F \)-closed property is equivalent to the quasi \( H \)-closed property.

2. Preliminaries

Throughout the present note, spaces always mean topological spaces on which no separation axioms are assumed. Let \( A \) be a subset of a space \( X \). The closure and the interior of \( A \) are denoted by \( \text{Cl}(A) \) and \( \text{Int}(A) \), respectively. A subset \( A \) is said to be \( \text{preopen} \) [9] (resp. \( \text{semi-open} \) [5], \( \alpha \)-open [10]) if \( A \subseteq \text{Int}(\text{Cl}(A)) \) (resp. \( A \subseteq \text{Cl}(\text{Int}(A)) \)). The complement of a preopen (resp. semi-open, \( \alpha \)-open) set is said to be \( \text{preclosed} \) (resp. \( \text{semi-closed} \), \( \alpha \)-closed). The intersection of all semi-closed sets of \( X \) containing \( A \) is called the \( \text{semi-closure} \) of \( A \) [3] and is denoted by \( s\text{Cl}(A) \). A subset \( A \) is said to be \( \text{feebly open} \) [6] if there exists an open set \( U \) of \( X \) such that \( U \subseteq A \subseteq s\text{Cl}(U) \). The following property is shown in [11, Lemma 3.1].

**Lemma 2.1.** A subset of a space \( X \) is \( \alpha \)-open in \( X \) if and only if it is semi-open and preopen in \( X \).

3. Quasi \( H \)-closed spaces
Lemma 3.1. If $A$ is a preopen set of a space $X$, then $\text{Cl} (\text{Int}(\text{Cl}(A))) = \text{Cl}(A)$ and $s\text{Cl}(A) = \text{Int}(\text{Cl}(A))$.

Proof. The first part is obvious and the second follows from $s\text{Cl}(A) = A \cup \text{Int}(\text{Cl}(A))$ [1, Theorem 1.5].

Lemma 3.2. A subset $A$ of a space $X$ is feebly open in $X$ if and only if it is $\alpha$-open in $X$.

Proof. Let $A$ be feebly open in $X$. There exists an open set $U$ of $X$ such that $U \subset A \subset s\text{Cl}(U)$. Since $U \subset \text{Int}(A)$ and $s\text{Cl}(U) = \text{Int}(\text{Cl}(U))$ by Lemma 3.1, we have $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ and hence $A$ is $\alpha$-open in $X$. Conversely, let $A$ be $\alpha$-open in $X$. We have $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ and hence $\text{Int}(A) \subset A \subset s\text{Cl}(\text{Int}(A))$. Therefore, $A$ is feebly open in $X$.

Definition 3.3. A space $X$ is said to be $F$-closed [2] if every feebly open cover of $X$ has a finite proximate subcover.

Theorem 3.4. The following are equivalent for a space $X$:

(a) $X$ is $F$-closed.
(b) $X$ is quasi $H$-closed.
(c) Every preopen cover of $X$ has a finite proximate subcover.
(d) For each family $\{F_\alpha|\alpha \in \nabla\}$ of preclosed sets in $X$ satisfying $\cap \{F_\alpha|\alpha \in \nabla\} = \emptyset$, there exists a finite subset $\nabla_0$ of $\nabla$ such that $\cap \{\text{Int}(F_\alpha)|\alpha \in \nabla_0\} = \emptyset$.

Proof. (a) $\Rightarrow$ (b) : The proof is obvious since every open set is feebly open.

(b) $\Rightarrow$ (c) : Let $\{V_\alpha|\alpha \in \nabla\}$ be a cover of $X$ by preopen sets of $X$. For each $\alpha \in \nabla$, $V_\alpha \subset \text{Int}(\text{Cl}(V_\alpha))$ and $\{\text{Int}(\text{Cl}(V_\alpha))|\alpha \in \nabla\}$ is an open cover of $X$. There exists a finite subset $\nabla_0$ of $\nabla$ such that $X = \cup \{\text{Cl}(\text{Int}(\text{Cl}(V_\alpha)))|\alpha \in \nabla_0\}$.

By Lemma 3.1, we obtain $X = \cup \{\text{Cl}(V_\alpha)|\alpha \in \nabla_0\}$.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) : These follow easily from Lemmas 2.1 and 3.2.

Remark 3.5. By Theorem 3.4, we observe that Example 4.1 of [2] is false since $\beta\mathbb{N} \times \beta\mathbb{N}$ is compact.

A subset $A$ of a space $X$ is said to be $F$-closed relative to $X$ [2] (resp. quasi $H$-closed relative to $X$ [12]) if for every cover $\{V_\alpha|\alpha \in \nabla\}$ of $A$ by feebly open (resp. open) sets of $X$, there exists a finite subset $\nabla_0$ of $\nabla$ such that $A \subset \cup \{\text{Cl}(V_\alpha)|\alpha \in \nabla_0\}$.
Theorem 3.6. A subset $A$ of a space $X$ is $F$-closed relative to $X$ if and only if it is quasi $H$-closed relative to $X$.

Proof. Suppose that $A$ is quasi $H$-closed relative to $X$. Let $\{V_\alpha|\alpha \in \mathcal{V}\}$ be a cover of $A$ by feebly open sets of $X$. By Lemma 3.2, $\{\text{Int}(\text{Cl}(\text{Int}(V_\alpha)))|\alpha \in \mathcal{V}\}$ is a cover of $A$ by open sets of $X$. There exists a finite subset $\mathcal{V}_0$ of $\mathcal{V}$ such that $A \subset \bigcup\{\text{Cl}(\text{Int}(V_\alpha))|\alpha \in \mathcal{V}_0\}$. By Lemmas 2.1 and 3.2, $V_\alpha$ is semi-open and hence $\text{Cl}(\text{Int}(V_\alpha)) = \text{Cl}(V_\alpha)$ for each $\alpha \in \mathcal{V}$. Therefore, we have $A \subset \bigcup\{\text{Cl}(V_\alpha)|\alpha \in \mathcal{V}_0\}$. The converse is obvious since every open set is feebly open.

It was pointed out in [12, p. 161] that every quasi $H$-closed subspace is quasi $H$-closed relative to the space but not conversely. In [2, Theorem 2.2], Chae and Lee showed that a feebly open subspace of a space $X$ is $F$-closed if and only if it is $F$-closed relative to $X$. The following theorem is a slight improvement of this result since every feebly open set is preopen.

Theorem 3.7. Let $A$ be a preopen set of a space $X$. The subspace $A$ is quasi $H$-closed if and only if $A$ is quasi $H$-closed relative to $X$.

Proof. Suppose that $A$ is preopen in $X$ and quasi $H$-closed relative to $X$. Let $\{V_\alpha|\alpha \in \mathcal{V}\}$ be a cover of $A$ by open sets of the subspace $A$. For each $\alpha \in \mathcal{V}$, there exists an open set $W_\alpha$ of $X$ such that $V_\alpha = W_\alpha \cap A$. Since $A$ is preopen in $X$, we have

$$V_\alpha \subset W_\alpha \cap \text{Int}(\text{Cl}(A)) = \text{Int}(W_\alpha \cap \text{Cl}(A)) \subset \text{Int}(\text{Cl}(W_\alpha \cap A)) = \text{Int}(\text{Cl}(V_\alpha)).$$

Therefore, $V_\alpha$ is preopen in $X$ and $\{\text{Int}(\text{Cl}(V_\alpha))|\alpha \in \mathcal{V}\}$ is a cover of $A$ by open sets of $X$. By Lemma 3.1, there exists a finite subset $\mathcal{V}_0$ of $\mathcal{V}$ such that $A \subset \bigcup\{\text{Cl}(V_\alpha)|\alpha \in \mathcal{V}_0\}$. Therefore, we obtain

$$A = \bigcup\{\text{Cl}(V_\alpha) \cap A|\alpha \in \mathcal{V}_0\} = \bigcup\{\text{Cl}_A(V_\alpha)|\alpha \in \mathcal{V}_0\},$$

where $\text{Cl}_A(V_\alpha)$ denotes the closure of $V_\alpha$ in the subspace $A$. This shows that $A$ is quasi $H$-closed.

A function $f : X \to Y$ is said to be $\alpha$-continuous [8] (resp. $\alpha$-irresolute [7]) if $f^{-1}(V)$ is $\alpha$-open in $X$ for every open (resp. $\alpha$-open) set $V$ of $Y$. By Lemma 3.2, $\alpha$-continuity (resp. $\alpha$-irresoluteness) is equivalent to feeble continuity (resp. feeble irresoluteness) due to Chae and Lee [2]. A function $f : X \to Y$ is said to be $\theta$-continuous [4] if for each $x \in X$ and each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(\text{Cl}(U)) \subset \text{Cl}(V)$.
Remark 3.8. For the properties on a function $f : X \to Y$, the following implications are known in [7] and [8]:

\[
\begin{align*}
\alpha - \text{irresoluteness} & \quad \Rightarrow \quad \alpha - \text{continuity} \Rightarrow \theta - \text{continuity}.
\end{align*}
\]

Lemma 3.9. If $f : X \to Y$ is $\theta$-continuous and $A$ is quasi $H$-closed relative to $X$, then $f(A)$ is quasi $H$-closed relative to $Y$.

Proof. The proof is obvious and is thus omitted.

Corollary 3.10 (Chae and Lee [2]). Let $X$ be an $F$-closed space and $f : X \to Y$ a function. Then, the following properties hold:

(a) If $f$ is a feebly continuous surjection, then $Y$ is quasi $H$-closed.
(b) If $f$ is a feebly irresolute surjection, then $Y$ is $F$-closed.
(c) If $f$ is feebly irresolute and $Y$ is Hausdorff, then $f(X)$ is closed in $Y$.

Proof. (a) and (b) are immediate consequences of Theorem 3.4 and Lemma 3.9. (c) follows from Theorem 3.4, Lemma 3.9 and the fact that if $B$ is quasi $H$-closed relative to $Y$ and $Y$ is Hausdorff then $B$ is closed in $Y$.

References


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