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A Class of Rank Tests For Comparing Several Treatments with a Control

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ABSTRACT

Consider a class of rank tests for comparing several treatments with a control and discuss some members among the class. New rank test based on orthogonal contrasts is proposed and compared with other well known tests. The approximate powers of the proposed test are also presented through the simulation studies.

1. Introduction

Let us consider an experimental situation where we examine many new treatments in hopes of improving on a control or a standard. This situations often arises, for example, in drug screening and the

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problem is to determine if any are significantly better than the control. A number of statistical procedures have been proposed for this problem:Bartholomew(1961) proposed the likelihood ratio test (LRT) and Robertson and Wright(1985) discussed rank analogue of LRT. On the other hand. Abelson and Tukey(1963) obtained the single contrast test by maximin principle and park(1990) explored classes of contrast tests based on ranks.

The single contrast test has been known that the test statistic is normally distributed with relatively easily computable mean and variance under both the null and alternative hypotheses. Furthermore, this test is somewhere most powerful test. This test, however, has very poor power in some alternative areas. On the other hand, the LRT maintains the reasonable power in entire alternative areas but it is pretty difficult to apply in real situations without special tables and nowhere most powerful. Robertson et. al(1988) advised that the contrast test is better than LRT when experimenters believes some effectiveness of treatments and numbers of treatments are small. Otherwise, the use of LRT was recommended.

To rectify this drawback of a single contrast test, Mukerjee, Robertson and Wright(1987) proposed the orthogonal contrast test(OCT) for use with samples from normal distributions. They compared the power of OCT with Ableson and Tukey(1963)'s single contrast test and LRT, and recommended the use of OCT, because this test has uniformly reasonable power over all the alternatives. Thus this paper considers an nonparametric extention of the OCT(say, orthogonal contrasts rank test(OCRT) and examine approximate powers through the simulation studies.

Ⅱ. A Class of Multiple contrast thests based on Ranks

We have (k+1) independent samples, $\{X_{ij},\ i=0,1,\cdots,k,\ j=1,2,\cdots,n_i\}$, from (K+1) populations. The i−th population has the continuous cumulative distribution function(CDF) F_i The population 0 is the control and the rest are the treatments. Let $F_0 \overset{st}{\leq} F_i$ mean $F_0(x) \geq F_i(x)$ for all $x \epsilon R$. It is of interest to test the null hypothesis

(2.1) $\qquad H_0^*: F_0 = F_i$, for $i=1,2,\cdots,k$

against the one−sided alternatives

(2.2) $\qquad H_1^*: F_0 \overset{st}{\leq} F_i$, for $i=1,2,\cdots,k$, and not H_0^*.

Frequently it is reasonable to assume the location model, namely,

(2.3) $\qquad F_i(x) = F(x-\theta_i),\ i=0,1,\cdots,k,$

Where F is an unknown continuous CDF with density f. This is a reasonable assumption in those cases where the treatments tend to affect the level of the response rather thatn the variability and the con trol is a placebo or no treatment. When this model holds, the hypotheses H_0^* and H_1^* cna be reexpressed as

(2.4) $H_0: \Theta_0 = \Theta_i,\ i=1,2,\cdots,k$, and

(2.5) $H_1: \Theta_0 \leq \Theta_i$, with at least one strict inequality, $i=1,2,\cdots,k$

To derive the OCRT for the case of $n_0=n_1=n_2=\cdots=n_k=n$, we follow the procedures suggested by Mukerjee, Robertson and Wright(1987). Let

$H_i' = H_1 \cap \{\Theta \epsilon R_{K+1}: \sum_{0\leq i\geq k} \Theta i = 0\}$. The cone H_i' is generated by

nonnegative multiples and convex combinations of the k corner vectors:

(2.6) $e_i = (e_{i0}, e_{i1}, \cdots, e_{ik})$

$= (-1, -1, \cdots, k, -1, \cdots, -1),\ i=1,2,\cdots,k$

with $e_{ii}=k$. The angle between any two corners is $COS^{-1}\{-(1/k)\}$.

And because of symmetry, H_1' has a unique center

(2.7) $\wedge = \sum_{1\leq i\geq k} e_i = (-k, 1, 1, \cdots, 1)$

and this vector makes equal angles with all the corners, ei. Let.

$a_i(r) = (a_{io}(r), a_{i1}(r), \cdots, a_{ir}(r))$ and

(2.8) $a_i(r) = r\wedge + (1-r)e_i$ for $i=1,2,\cdots,k$

where $r\epsilon[0,1]$

Let R_{ij} be ranks of X_{ij} and $\bar{R}_i$ be the average of ranks of the i－th treatment, $\bar{R}$ be the average of total ranks$(N=(k+1)n)$. Then we can define statistics

(2.9) $$T_i(r) = \left(\frac{12\lambda}{N+1}\right) 1/2\ \frac{\sum_{j=0}^{k} a_{ij}(r)\bar{R}_j}{[\sum_j a_{ij}(r)^2]^{1/2}} \quad , \text{ for } i=1,2,\cdots,k,$$

where $\lambda = n/N$, assumed λ is being constant as $N\rightarrow\infty$.

Now we can propose a class of tests based on Ti(r)'s for H_0 against H_1 is

(2.10) $T = \max_{1\leq i\leq k} T_i(r)$

and the test based on (2,10) rejects H_0 for large values of T.

Remark 2.1: From (2.8) if $r=1$, we can easily see that T does not depend on subscripts i and becomes single contrast test statistics based on ranks.

And if $r=\frac{1}{2}$, then T becomes $\max_{1\leq i\leq k}(\bar{R}_i - \bar{R}_0)$, which is a rank analogue of Dunnett's test statistics.

Let us set Ni(r) to be the numberator of $T_i(r)$. Let $\bar{R}_T = \sum_{1\leq j\leq k} \bar{R}_j/k$ and $\bar{R}_{-i} = \{\bar{R}_0 + \sum_{(1\leq j\leq k, j\neq i)} \bar{R}_j\}/(k)$. Using (2.6), (2.7) and (2.8) after some simple algebra.

$$N_i(r) = rk(\bar{R}_T - \bar{R}_0) + (1-r)(k)\ (\bar{R}_i - \bar{R}_{-i}).$$

Now$(\bar{R}_T-\bar{R}_0)$ measures the amount by which the average of the all treatments ranks differ from the average of the control ranks, and
$(\bar{R}_i-\bar{R}_{-i})$ measures the amount by which the average of the i-th treatment ranks differ from the weighted average of the rest of the ranks. Thus
$N_i(r)$ is a multiple of the weighted average of $(\bar{R}_T-\bar{R}_0)$ and $(\bar{R}_i-\bar{R}_{-i})$, where the weights depends on r. Thus we can construct some test statistics depending on r.

Ⅲ. Derivation of OCRT

Let$T(r)=(T_1(r), T_2(r),\cdots,T_K(r))$, Under the null hypothesis, the distribution of $(T(r))$ is asymptotically normal distribution with mean vector 0 and variance-covariance matrix $\sigma^*\Sigma$, where $\Sigma_{ii}=1$ and

(3.1) $$\Sigma_{ii'}=\frac{\Sigma_{0\le j\le k}\ a_{ij}(r)a_{i'j}(r)}{[\Sigma_j a_{ij}(r)^2]^{1/2}\ [\Sigma_j a_{i'j}(r)^2]^{1/2}}$$

for $i\neq i'=1,2,\cdots,k$ and $\sigma^*=\frac{N(N+1)}{12n}$.

Following Mukerjee, Robertson, and Wright(1987), we propose an orthogonal contrast rank test (OCRT) which is based on T(r), where the value of r is chosen such that the covariance in (3.1) are equal to zero.

This value of r, say r0, is the unique solution in [0, 1] of the equation

(3.2) $$(K-3)r^2+4r-1=0$$

and is given by

(3.3) $$r_0=\frac{1}{4} \qquad \text{if}(K-2)=1,$$

$$=\{-2+(K+1)^{1/2}\}/(k-3), \quad \text{otherwise.}$$

Thus we propose the OCRT using this r_0 by

(3.4) $$T_1=\max_{1\le i\le k} T_i(r_0);$$

and the test rejects H_0 for large values or T_1.

Under H_0, the statistics $T_i(r_0)$ are asymptotically N(0,1) variables and are uncorrelated. Thus an approximate size α test of H_0 is

(3.7) $$\text{reject } H_0 \text{ if } T_1> \Phi^{-1}((1-\alpha)^{1/k}).$$

This competes the OCRT testing procedures.

IV. Illustrated example

To understand OCRT better, let us consider following the hypothetical example. A researcher is very much interested in effectiveness of several treatments reducing the blood pressure, The following data are measured the reduction of blood pressure of 60 experimental rats in 10 minites after electrical shocks and treatments are given. The first one serves as the control;that is, no treatment are given. And the rest ones are the treatments depending on dosages of drugs.
Assume that all rats are very similar conditions, a randomization is applied to divide four experimental groups.

TREATMENTS

Control	#1	#2	#3
15.21	.45	1.87	5.77
8.79	27.16	13.56	22.08
31.27	4.56	27.85	8.00
9.13	2.81	2.27	9.23
5.32	2.51	33.39	21.63
5.84	24.19	3.39	1.91
12.31	7.42	5.62	14.42
4.96	2.20	41.90	37.08
5.07	68.25	37.90	20.28
8.00	61.90	.35	29.08
18.83	20.47	24.14	2.54
.32	15.07	17.32	11.75
4.67	1.49	9.36	1.25
1.75	38.43	18.09	5.69

Form these data we can test the hypothesis $H_0:\theta_0=\theta_1=\theta_2=\theta_3$ against $H_1:\theta_0\leq\theta_i$, with at least one strict inequality, $i=1,2,\cdots 3$. When $k=3$, the orthogonal contrasts are $(-3, 5, -1, -1)$, $(-3, -1, 5, -1)$ and $(-3, -1, -1, 5)$. Now we can evaluate the numerators of $T_i(r_0)$'s are -16.933, 55.067 and -18.533 and $T_1=3.89$. On the other hand, the single contrast(T_2. T for the case of $r=1$) is $(-3,1,1,1)$ so that $T_2=.421725$. Thus the p-value of the test based on T_1 is less than 0.01 and the P-value of the test based on T_2 is about 0.3372. From these observations we can conclude H_1 by the test based on OCRT, but we can reject H_0 by the test based on the single contrast test in this particular example.

V. Approximate power comparisons in small samples

A simulation study is undertaken to compare the performance of the OCRT proposed here. The tes. statistics studied are OCRT(T_1), single contrast test(T_2), Robertson and Wright(1985)'s LRT based on ranks(T_3).

We use the equal sample size configurations: $n_0 = n_1 = \cdots = n_k = n$.
The n value is taken as 10 and the approximate 0.05 critical values of these statistics are used.

The distributions, used to generate the samples, are Cauchy, exponential, lognormal and normal. The scale parameters for all these distributions are given as 1. Subr outines RNCHY, RNEXP, RNLNL of the IMSL are used to generate the samples. The study includes 3 values for the nunber of treatments, namely 2,3 and 4. The size and the power of each test are estimated suing 10,000 replications. The power estimates for various parameters configurations are in table 1.

Table 1

(k=2)

Normal distribution

Location parameters			Test statistics		
θ_0	θ_1	θ_2	T_1	T_2	T_3
0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.22	0.09	0.21
0.0	0.5	0.5	0.23	0.23	0.19
0.0	0.5	1.0	0.56	0.45	0.58
0.0	1.0	1.0	0.62	0.70	0.66
0.0	1.0	1.5	0.85	0.88	0.90
0.0	1.5	1.5	0.91	0.97	0.95

Cauchy distribution

Location parameters			Test statistics		
θ_0	θ_1	θ_2	T_1	T_2	T_3
0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.11	0.08	0.10
0.0	0.5	0.0	0.14	0.15	0.14
0.0	0.5	1.0	0.24	0.24	0.23
0.0	1.0	1.0	0.28	0.36	0.29
0.0	1.0	1.5	0.39	0.46	0.40
0.0	1.5	1.5	0.44	0.58	0.47

Exponential distribution

Location parameters			Test statistics		
θ_0	θ_1	θ_2	T_1	T_2	T_3
1.0	1.0	1.0	0.05	0.05	0.05
1.0	1.5	1.0	0.15	0.11	0.14
1.0	1.5	1.5	0.17	0.22	0.17
1.0	2.0	1.5	0.27	0.32	0.28
1.0	2.0	2.0	0.33	0.44	0.36
1.0	2.0	3.0	0.53	0.60	0.56

Lognormal distion

Location parameters			Test statistics		
θ_0	θ_1	θ_2	T_1	T_2	T_3
0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.5	0.0	0.22	0.15	0.22
0.0	0.5	0.5	0.24	0.33	0.26
0.0	0.5	1.0	0.55	0.56	0.56
0.0	1.0	1.0	0.62	0.79	0.68
0.0	1.0	1.5	0.86	0.92	0.89

(K=3)

Normal distribution

Location parameters				Test statistics		
θ_0	θ_1	θ_2	θ_3	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.5	0.0	0.0	0.20	0.09	0.20
0.0	0.5	0.0	0.5	0.23	0.16	0.26
0.0	0.5	0.5	0.5	0.22	0.30	0.24
0.0	0.5	1.0	0.5	0.48	0.46	0.50
0.0	1.0	1.0	1.0	0.54	0.79	0.69

Cauchy distribution

Location parameters				Test statistics		
θ_0	θ_1	θ_2	θ_3	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.0	0.5	0.10	0.08	0.11
0.0	0.5	0.0	0.5	0.12	0.12	0.10
0.0	0.5	0.5	0.5	0.12	0.19	0.13
0.0	0.5	1.0	0.5	0.20	0.21	0.21
0.0	0.5	1.0	1.0	0.24	0.28	0.26
0.0	1.0	1.0	1.0	0.24	0.36	0.28

Exponential distribution

Location parameters				Test statistics		
θ_0	θ_1	θ_2	θ_3	T_1	T_2	T_3
1.0	1.0	1.0	1.0	0.05	0.05	0.05
1.0	1.0	1.5	1.0	0.13	0.08	0.13
1.0	1.5	1.5	1.0	0.16	0.14	0.17
1.0	1.5	1.5	1.5	0.15	0.22	0.16
1.0	1.5	2.0	2.0	0.28	0.36	0.30
1.0	1.5	2.0	2.5	0.37	0.43	0.39

Lognormal distribution

Location parameters				Test statistics		
θ_0	θ_1	θ_2	θ_3	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.0	0.20	0.11	0.19
0.0	0.5	0.0	0.5	0.22	0.22	0.26
0.0	0.5	0.5	0.5	0.21	0.37	0.24
0.0	0.5	1.0	0.5	0.49	0.60	0.51
0.0	0.5	1.0	1.0	0.55	0.69	0.63

(K=4)

Normal distribution

Location parameters					Test statistics		
θ_0	θ_1	θ_2	θ_3	θ_4	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.0	0.0	0.18	0.09	0.17
0.0	0.5	0.0	0.5	0.0	0.23	0.16	0.26
0.0	0.5	0.5	0.5	0.0	0.23	0.26	0.28
0.0	0.5	0.5	0.5	0.5	0.20	0.40	0.25
0.0	0.5	1.0	1.0	0.5	0.60	0.65	0.62

Cauchy distribution

Location parameters					Test statistics		
θ_0	θ_1	θ_2	θ_3	θ_4	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.0	0.0	0.09	0.07	0.08
0.0	0.5	0.0	0.5	0.0	0.11	0.09	0.11
0.0	0.5	0.5	0.5	0.0	0.12	0.13	0.12
0.0	0.5	0.5	0.5	0.5	0.11	0.18	0.13
0.0	0.5	1.0	1.0	0.5	0.21	0.30	0.23

Exponential distribution

Location parameters					Test statistics		
θ_0	θ_1	θ_2	θ_3	θ_4	T_1	T_2	T_3
1.0	1.0	1.0	1.0	1.0	0.05	0.05	0.05
1.0	1.0	1.5	1.0	1.0	0.12	0.08	0.11
1.0	1.5	1.0	1.5	1.0	0.15	0.12	0.16
1.0	1.5	1.5	1.5	1.0	0.15	0.17	0.18
1.0	1.5	1.5	1.5	1.5	0.14	0.24	0.16
1.0	1.5	2.0	2.0	1.5	0.24	0.36	0.28

Lognormal distribution

Location parameters					Test statistics		
θ_0	θ_1	θ_2	θ_3	θ_4	T_1	T_2	T_3
0.0	0.0	0.0	0.0	0.0	0.05	0.05	0.05
0.0	0.0	0.5	0.0	0.0	0.18	0.10	0.17
0.0	0.5	0.0	0.5	0.0	0.22	0.17	0.16
0.0	0.5	0.5	0.5	0.0	0.21	0.26	0.25
0.0	0.5	0.5	0.5	0.5	0.19	0.39	0.28
0.0	0.5	1.0	1.0	0.5	0.52	0.65	0.73

Ⅳ. Conclusion

We have derived the OCRT for comparing several treatments with a control and examined approximate powers of it with two other well known tests, single contrast test and LRT through a simulation study. As Mukerjee, Robertson and Wright(1987) pointed out that OCT maintained a uniformly reasonable power in normal case, this limited simulation study shows that OCRT also has the good power in various distributions for the case of the wicoxon scores. Especially, we find that OCRT is verypowerful in the slippage configuration of the parameters and has very similar powers comparing with LRT in mang cases. Since a importaint shortcomings of the single contrast test has very poor power in the slippage configurations. this approximate critical values easily comparing with LRT, we can obtain the approximate critical values easily comparing with LRT, we would better consider this testing procedure for this kind of partially ordered restrictions.

References

Apelson, R.P. and Tukey, J.W.(1963):Efficient utilization of non-numerical information in quantitive analysis:General theory and the case of simple order, Ann, Math. Statist., 34, 1347-69.

Bartholomew, D.J.(1961):A test of homogeneity of means under restricted alternatives. J.Roy. Statist. Soc. B. 23. 239-281.

Dunnett, C.W.(1955):A multiple comparisons procedure for comparing several treatments with a control. J. Amer, Statist. Assoc. 50, 1096-1121.

Mukerjee, H., Robertson, T. and wright, F.T.(1987):Comparison of several treatments with a control using multiple contrasts. J. Amer, Statist. Assoc., 82, 902-910.

Park, S.(1990):Nonparametric test for the simple tree alternatives. Ph.D. Dissertation, SUNY-Buffalo.

Robertson, T. and Wright, F.T.(1985):One-sided comparisons for treatments with control. Canadian J. Statist., 13, 109-122.

Robertson, T. and wright, F.T. and Dykstra, R.L.(1988):Order restricted statistical inference. John Wiley & Sons.