Journal of the KSQC
Vol. 19, No. 2, Dec.1991

1991년 12월 韓國品質管理學會誌 제19권 제2호

A Study for Test for NBU in Convex Ordering

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ABSTRACT

In this paper, we suggest the test statistic based on the total time on test (TTT-) transtorm for testing exponentiality against new better than used in convex ordering (NBUC) and its dual, new worse than used in convex ordering (NWUC). The validity of the test is examined by simulation for some alternative distributions when the sample size is n = 10 and n = 20.

1. Introduction

In performing reliability analysis, it has been founded very useful to classify life distributions using the concept of stochastic odering. For definitions of several calsses of life distributions, e.g., IFR, IFRA, NBU, NBUE, DMRL and their duals, see Bryson and Siddiqui(1969) and Barlow and Proschan(1975). A new class of life distributuons, namely new better than used in convex odering (NBUC) and its dual, new worse than in convex odering (NWUC), are introduced by Cao and Wang(1991).

Let X be a non-negative random variable representing equipment life with distribution $F(t)$ and survival function $\bar{F}(t) = 1 - F(t)$ and X_t be residval life of the equipment of age t with distribution

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$F_t(x)$ and survival function

$$\bar{F}t(x) = \begin{cases} \dfrac{\bar{F}(x+t)}{\bar{F}(t)}, & \text{if } \bar{F}(t) > 0; \\ 0, & \text{if } \bar{F}(t) = 0. \end{cases}$$

It is well known that F belongs to the IFR (DFR) class if and only if X_t is decreasing (increasing) in $t \geq 0$ in stochastic ordering. Cao and Wang defined a new kind of class of life distributions : F is NBUC (NWUC) if and only if X_t is smaller (larger) than X for any $t \geq 0$ in convex ordering. Namely, $X \in$ NBUC (NWUC) if and only if

$$\int_x^\infty \bar{F}(t+y)dy \leq (\geq)\ \bar{F}(t) \cdot \int_x^\infty \bar{F}(y)dy, \quad x, t \geq 0$$

The sclaed total time on test (TTT-) transform $\varphi_F(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx/\mu$, where $\mu = \int_0^\infty \bar{F}(x)dx < \infty$, and TTT-plot were introduced by Barlow and Campo(1975) as a tool in the statistical analysis of life time data. In Section 2, we shall briefly present the TTT-transform and the TTT plot and also correspondence between the aging property NBUC and the scaled TTT-transform. This correspondence will be used in Section 3 to get idea for the test statistic for testing H_0 : F is the exponential distribution against H_1 : F is NBUC but not exponential. In section 4, simulation results are discussed.

2. The Total Time on Test Concept.

Let F be a life distribution with survival function $\bar{F}(t) = 1 - F(t)$ and finite mean $\mu = \int_x^\infty \bar{F}(y)dx$. The scaled TTT-transform φ_F of F is defined by

$$\varphi_F(t) = \frac{1}{\mu}\int_0^{F^{-1}(t)} \bar{F}(u)du \qquad \text{for } 0 \leq t \leq 1,$$

where $F^{-1}(t) = inf\{\, x : F(x) \geq t \,\}$.

In particular we note that if F is the exponential distribution then the scaled TTT-transform is given by

$$\varphi_F(t) = t \quad 0 \leq t \leq 1.$$

Scaled TTT-transform for some families of life distributions are presented by Barlow(1979) and Bergman(1979). For further details about the TTT-transform see Barlow and Campo(1975), Barlow (1979), Bergman(1979) and Klefsjö(1982).

Assume that $t_{(1)} \le t_{(2)} \le \cdots \le t_{(n)}$ is an ordered sample from a life distribution F (and let $t_0 = 0$). Further let

$$D_j = (n-j-1)\,(t_{(j)} - t_{(j-1)}) \qquad \text{for } j = 1, 2, \cdots, n$$

denote the normalized spacings and let

$$S_j = \sum_{k=1}^{j} D_k \qquad \text{for } j = 1, 2, \cdots, n$$

(and $S_0 = 0$) denote the total time on test at t_j

A natural choice of estimator of the scaled TTT-transform is empirical scaled TTT-transform

$$\varphi F_n(t) = \frac{H_n^{-1}(t)}{H_n^{-1}(1)} \quad \text{for } 0 \le t \le 1,$$

where

$$H_n^{-1}(t) = \int_0^{F_n^{-1}(t)} \bar{F}_n(s)\, ds \quad \text{for } 0 \le t \le 1,$$

F_n is the empirical distribution function and $\bar{F}_n = 1 - F_n$. Calculations show that

$$H_n^{-1}\left(\frac{j}{n}\right) = \frac{S_j}{n} \text{for } j = 0, 1, \cdots, n.$$

By using

$$u_j = \frac{S_j}{S_n} \text{for } j = 0, 1, \cdots, n$$

we get that

$$\varphi F_n\left(\frac{j}{n}\right) = u_j \qquad \text{for } j = 0, 1, \cdots, n.$$

The TTT-plot is obtained by plotting $u_j = S_j / S_n$ against j/n for $j = 0, 1, \cdots, n$ and then connecting the plotted points by straight lines.

As a consequence of the Glivenko-Cantelli Lemma we get that if F is strictly increasing then

$$\varphi F_n\left(\frac{j}{n}\right) \to \varphi F(t)$$

with probability one and uniformly in [0, 1] when $n \to \infty$ and $j/n \to t$ (Barlow et al. (1972), p.239). Because of this, Barlow and Campo(1975) suggested a comparison of the TTT-plot with graphs of scaled TTT-transforms for making model identification.

NBUC can be translated to property of the scaled TTT-transform. We shall here only suggest the relationship which will be used in the next section to get idea for the test statistic.

Theorem A life distribution F is NBUC (NWUC) if and only if

$$\frac{\varphi_F(F(x+t)) - \varphi_F(F(x))}{1 - \varphi_F(F(x))} \geq (\leq)\ F(t), \qquad x, t \geq 0$$

Proof By the definition of NBUC (NWUC),

$$X \in NBUC(NWUC) \Leftrightarrow \int_x^\infty F(t+y)dy \leq (\geq)\ F(t) \cdot \int_x^\infty F(y)dy,\ x, t \geq 0$$

$$\Leftrightarrow \int_{t+x}^\infty F(y)dy \leq (\geq)\ F(t) \cdot \int_x^\infty F(y)dy,\ x, t \geq 0$$

$$\Leftrightarrow \mu - \int_0^{t+x} F(y)dy \leq (\geq)\ F(t) \cdot (\mu - \int_0^x F(y)dy),\ x, t \geq 0$$

$$\Leftrightarrow F(t) \leq (\geq) \frac{1}{\mu}\int_0^{t+x} F(y)dy - F(t) \cdot \frac{1}{\mu}\int^x F(y)dy,\ x, t \geq 0$$

Since $\varphi_F(F(x)) = \frac{1}{\mu}\int_0^x F(y)dy$,

$$\Leftrightarrow F(t) \leq (\geq)\ \varphi_F(F(t+x)) - F(t) \cdot \varphi_F(F(x)),\ x, t \geq 0$$

$$\Leftrightarrow F(t) \leq (\geq) \frac{\varphi_F(F(t+x)) - \varphi_F(F(x))}{1 - \varphi_F(F(x))}, \qquad x, t \geq 0$$

Hence the theorem is proved.

3. Test Statistic based on the Scaled TTT-transform.

In this section we shall suggest the test statistic for testing exponentiality against the NBUC. We know from Theorem that life distribution F is NBUC (NWUC) if and only if

$$\frac{\varphi_F(F(t+x)) - \varphi_F(F(x))}{1 - \varphi_F(F(x))} \geq (\leq)\ F(t)$$

This means that

$$\frac{\varphi_{F_n}(F_n(t_{(i)}+t_{(j)})) - \varphi_{F_n}(F_n(t_{(i)}))}{1 - \varphi_{F_n}(F_n(t_{(i)}))} \geq (\leq)\ F_n(t_{(j)}) \qquad (3.1)$$

where $F_n(\cdot)$ is the empirical distribution of $F(\cdot)$. By the definition of the empirical TTT-transform

$$(3.1) \Leftrightarrow \frac{\varphi_{F_n}(F_n(t_{(i)}+t_{(j)}))-u_i}{1-u_i} \geq (\leq) \frac{j}{n} \qquad (3.2)$$

If $t_{(k)} \leq t_{(i)} + t_{(j)} < t_{(k+1)}$, then

$$\varphi_F(F_n(t_{(i)}+t_{(j)})) = u_k + (1-\frac{k}{n}) \frac{(t_{(i)}+t_{(j)}-t_{(k)})}{\bar{t}}$$

where $t = \sum^{n}_{i=1} t_{(i)}/n$.

If $t_{(n)} \leq t_{(i)} + t_{(j)}$, then

$$\varphi_{F_n}(F_n(t_{(i)}+t_{(j)})) = u_n = 1$$

Therefore

$$(3.2) \Leftrightarrow \begin{cases} u_k - u_i + (1-\frac{k}{n}) \frac{(t_{(i)}+t_{(j)}-t_{(k)})}{t} \geq (\leq) \frac{j}{n}(1-u_i), \\ \qquad \text{if } t_{(k)} \leq t_{(i)}+t_{(j)} \leq t_{(k+1)} \text{ and } k \leq n-1 \\ (1-u_i)(1-\frac{j}{n}) \geq (\leq) 0, \qquad \text{if } t_{(i)}+t_{(j)} \geq t_{(n)}. \end{cases} \qquad (3.3)$$

From conditioon (3.3), We can define T_{ij} as follows ;

$$T_{ij} = \begin{cases} u_k - u_i + (1-\frac{k}{n}) \frac{(t_{(i)}+t_{(j)}-t_{(k)})}{t} - (\frac{j}{n})(1-u_i) \\ \qquad \text{if } t_{(k)} \leq t_{(i)}+t_{(j)} \leq t_{(k+1)} \text{ and } k \leq n-1 \\ (1-u_i)(1-\frac{j}{n}) \qquad \text{if } t_{(i)}+t_{(j)} \geq t_{(n)}. \end{cases}$$

And summation over i and j gives the test statistc

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij}$$

We expect a positive (negative) value of T if F is NBUC (NWUC), but not exponential.

4. Simulation Results and Conclusion

In section 3, we obtained the test statistic T for testing exponentiality against the NBUC. But we can not obtain the simple expression of T, Since the value of $F_n(X_{(i)}+X_{(j)})$ depends on the magnitudes of $X_{(i)}$ and $X_{(j)}$. Therefore we can not obtain the asymptotic distribution of the test statistic T, its consistency and the asymptotic effciency.

Hence we show the validity of the test statistic by simulation when the sample size is n = 10 and n = 20. For the simulation, we used subroutines in IMSL (CYBER 962−31) to generate random numbers from exponential, Weibull and gamma distribution given respectively by

$$F_1(x) = 1 - exp(-\theta x) \qquad \text{for } x \geq 0,\ \theta \geq 0$$
$$F_2(x) = 1 - exp(-x\ ) \qquad \text{for } x \geq 0,\ \theta \geq 0$$
$$F_3(x) = \frac{1}{\Gamma(\theta)} \int_0^x t^{\theta-1} e^{-t}\, dt \qquad \text{for } x \geq 0,\ \theta \geq 0$$

In the simulation study, we compare the mean value of T in exponential distribution and alternative distributions. 10000 replcations were performed for each distribution. Simulation results are summerized in Table 1.

Table 1 shows following facts ;

1) In Weibull distribution

if $\theta < 1$, Weibull distribution $\in$ NWUC
if $\theta > 1$, Weibull distribution $\in$ NBUC

2) In gamma distribution

if $\theta < 1$, gamma distribution $\in$ NWUC
if $\theta > 1$, gamma distribution $\in$ NBUC

These are of course expected. Therefore we note that the test statistic T for NBUC is useful.

Table 1. Empirical mean of T based on 10000 replications

Distribution	Sample size n = 10	Sample size n = 20
Exponential $\theta = 1$	.77607	−1.76783
Gamma $\theta = 0.5$	−8.68471	−36.49891
Gamma $\theta = 0.8$	−3.05317	−11.71513
Gamma $\theta = 1.5$	2.64932	13.36678
Gamma $\theta = 2.0$	4.51834	21.49049
Weibull $\theta = 0.5$	−15.48802	−70.66370
Weibull $\theta = 0.8$	−5.06394	−21.36792
Weibull $\theta = 1.5$	4.79640	22.92285
Weibull $\theta = 2.0$	6.75866	31.20948

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