ON THE BIFURCATION OF SUBHARMONIC ORBITS 
FOR GENERAL MAPS AT STRONG RESONANCES 

YONG IN KIM 

1. Introduction 
This paper is concerned with the generalization of the results given 
by Kim and Lee ([12]) for a typical one-parameter family of area-

preserving maps, so called Henon maps, to a general one-parameter fam-

ily of maps at strong resonances. 

The analysis for the bifurcation of the n-cycles at strong resonances 
(i.e., n = 3, 4) in this general case starts with imposing the assumptions 
that the complex conjugate eigenvalues of the linear part of a map lie 
on the unit circle in the complex plane and move along the unit circle 
as the parameter varies through zero. 

To investigate the occurrence of subharmonic orbits from the origin 
for a general one-parameter family of maps, the theory of normal forms 
and the method of Liapunov-Schmidt reduction are also used here, but 
treated only briefly in Section 2 by referring the interested readers to 
the previous work ([12]) for more details. 

The actual analysis and calculation in Section 3 and 4 employed 
to reveal the bifurcation pattern of the n-cycles (n = 3, 4) in this 
general case yield many notable results, which, of course, should cover 
the previous results ([12]) obtained for a typical Henon map. 

2. Preliminaries 
Consider a general one-parameter family of maps on \( \mathbb{R}^2 \) 

(2.1) \[ F_\mu : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \]

where \( F_\mu \in C^\infty \) and \( \mu \) is a real parameter. We may assume that 
\( F_\mu (0) = 0 \) for any \( \mu \in \mathbb{R} \). 

Received September 21, 1992
Let \( D_z F_\mu(0) = A_\mu \in \mathbb{R}^{2 \times 2} \) and \( \lambda(\mu) \), \( \bar{\lambda}(\mu) \) be eigenvalues of \( A_\mu \) for \( \mu \) sufficiently small and let \( \lambda_0 = \lambda(0) \), \( \bar{\lambda}_0 = \bar{\lambda}(0) \) be eigenvalues of \( A_0 \).

We assume that

\[
(2.2) \quad |\lambda(\mu)| = 1, \quad \lambda_0 \neq \pm 1
\]
\[
(2.3) \quad \frac{d}{d\mu} \arg \lambda(\mu)|_{\mu=0} > 0.
\]

Notice that the condition (2.3) implies that the eigenvalues of the linear part of \( F_\mu \) move along the unit circle as \( \mu \) varies through 0.

Since \( F \in C^\infty \), we can write

\[
(2.4) \quad \lambda(\mu) = \lambda_0(1 + \lambda_1 \mu + O(|\mu|^2)).
\]

From (2.2) and (2.3), we can write

\[
(2.5) \quad \lambda_1 = 2\pi i a (a > 0), \quad \lambda_0 = \exp(2\pi i \theta_0) \quad (\theta_0 \neq 0, 1/2 \ (\text{mod } 1))
\]

and

\[
(2.6) \quad \lambda(\mu) = \lambda_0 e^{2\pi i a \mu + O(|\mu|^2)}.
\]

By letting \( z = x_1 + ix_2 \), we can rewrite the given real map (2.1) in the following complex form

\[
(2.7) \quad z' = F_\mu(z) = \lambda(\mu)z + \sum_{l \geq 2} R_l(\mu, z, \bar{z}),
\]

where

\[
R_l(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, \quad l \geq 2.
\]

Now, we put (2.7) in a normal form by successive applications of a \( \mu \)-dependent change of variables of the following form

\[
(2.8) \quad z = w + \psi_l(\mu, w, \bar{w}) \equiv T_l(\mu, w), \quad l \geq 2,
\]
where

$$\psi_l(\mu, w, \bar{w}) = \sum_{p+q=l} \gamma_{pq} w^p \bar{w}^q, \quad l \geq 2,$$

with a suitable choice of the coefficients $\gamma_{pq}(\mu)$. According to the theory of normal forms for maps ([4,6,7,8]), we can transform the given map (2.7) to the normal forms given in Kim and Lee ([12]) (Refer to Lemma 1 in [12]).

Now, following the method used in Kim ([12]), we can reduce the study of the occurrence of $n$-cycles into that of finding zeros of an algebraic function (so called bifurcation function) as stated in the following Lemma (For the proof, refer to the Lemma 2 in Kim ([12])).

**Lemma 1.** Assume that $\lambda_0^n = 1(n \geq 3)$ and let $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$ be a $n$-cycle of the map $F\mu$ given in normal form. Let $S$ be a right-shift operator $(x_1, \cdots, x_{n-1}, x_n) \rightarrow (x_2, \cdots, x_n, x_1)$ and $\mathcal{F}_\mu(x) = (F_\mu(x_1), \cdots, F_\mu(x_n))$. Let $y = Px, y = (y_1, \cdots, y_n) \in \mathbb{C}^n$, where each column of $P$ consists of eigenvectors of $S$. Define a map $\Phi : \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}^n$ by $\Phi(y, \mu) = P \mathcal{F}_\mu(P^{-1}y) - \Lambda y$, where $\Lambda = \text{diag}(1, \lambda_0, \cdots, \lambda_0^{n-1})$. Let $L = D_y \Phi(0, 0)$ and write $y = y_n v_n + w$, where $v_n = (0, \cdots, 0, 1) \in \text{Ker} L$ and $w \in \text{Im} L$. Let $E : \mathbb{C}^n \rightarrow \text{Im} L$ be a projection. Let $z = \frac{1}{n} y_n$.

Then finding the $n$-cycle $(x_1, \cdots, x_n)$ of $F\mu$ is equivalent to solving the following equation in $\mathbb{C}$:

$$\lambda_0 z = F_\mu(z) = \lambda(\mu) z + R(\mu, z, \bar{z}). \quad (2.9)$$

Moreover, if we write

$$x_1 \equiv \phi(\mu)(z) \equiv z + \frac{1}{n} \sum_{j=1}^{n-1} w_j^*(n z, \mu), \quad (2.10)$$

where $w^* = (w_1^*, \cdots, w_{n-1}^*)$ satisfies the equation

$$E \Phi(y_n v_n + w^*(y_n, \mu)) = 0$$

then the other $n$-periodic points $x_2, \cdots, x_n$ are given by

$$x_j = \phi(\mu(\lambda_0^{j-1} z)) (j = 2, \cdots, n). \quad (2.11)$$
3. Bifurcation analysis of 3-cycles

(i) The case $n = 3$ and $c_{02}(0) \neq 0$.

In this case, $\lambda_0 = e^{2\pi i/3}$ and $F_\mu(z)$ has the normal form

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)z^2 + \mathcal{O}(|z|^3)$$

where $\lambda(\mu) = \lambda_0(1 + \mu \lambda_1 + \mathcal{O}(|\mu|^2))$ with $\lambda_1 = 2\pi i a$ ($a > 0$). Then (2.9) becomes

$$\mu_1 z + \bar{\lambda}_0 c_{02}(0)\bar{z}^2 + \mathcal{O}(|\mu|^2 |z| + |\mu||z|^2 + |z|^3) = 0.$$  

Letting $z = re^{2\pi i \phi}$ and separating the trivial solution $r = 0$, we have

$$2\pi i a \mu + \bar{\lambda}_0 c_{02}(0)re^{-6\pi i \phi} + g(\mu, r, \phi) = 0,$$

where $g(\mu, r, \phi) = \mathcal{O}(|\mu|^2 + |\mu| r + r^2)$ and $g(\mu, r, \phi + 1/3) = g(\mu, r, \phi)$.

Set

$$r = 2\pi a \cdot \left| \frac{\mu}{c_{02}(0)} \right| \cdot (1 + r_1),$$

$$\phi = \phi_0 + \phi_1,$$

$$\phi_0 = -\frac{1}{36} - \frac{1}{6\pi} \arg \mu + \frac{1}{6\pi} \arg c_{02}(0) \pmod{1/3}.$$

Substituting (3.4) in (3.3) and simplifying, we have

$$1 - e^{-6\pi i \phi_1} (1 + r_1) + g_2(\mu, r_1, \phi_1) = 0,$$

where

$$g_2(\mu, r_1, \phi_1) = (2\pi i a \mu)^{-1} g(\mu, 2\pi a \left| \frac{\mu}{c_{02}(0)} \right| (1 + r_1), \phi_0 + \phi_1) = \mathcal{O}(|\mu|).$$

Let

$$h(\mu, r_1, \phi_1) = 1 - e^{-6\pi i \phi_1} (1 + r_1) + g_2(\mu, r_1, \phi_1).$$

By the implicit function theorem, we have

$$r_1 = r_1(\mu), r_1(0) = 0, \phi_1 = \phi_1(\mu), \phi_1(0) = 0.$$
Consequently, we have from (3.4),

\[ r = 2\pi a \cdot \left| \frac{\mu}{c_{02}(0)} \right| \cdot (1 + O(|\mu|)) = 2\pi a \left| \frac{\mu}{c_{02}(0)} \right| + O(|\mu|^2), \]

and the coordinates of the 3-periodic points for the area-preserving map \( F_\mu(z) \) in normal form are given, from (2.10), (2.11) and (3.5), by

\[
\begin{align*}
 x_1 &= \phi_\mu(z) \equiv z(\mu) + O(|\mu|^2) \\
 &= r(\mu)e^{2\pi i \phi(\mu)} + O(|\mu|^2) = 2\pi a \left| \frac{\mu}{c_{02}(0)} \right| e^{2\pi i \phi_0} + O(|\mu|^2), \\
 x_2 &= \phi_\mu(\lambda_0 z), \\
 x_3 &= \phi_\mu(\lambda_0^2 z).
\end{align*}
\]

Note that as \( \mu \) varies from \( \mu < 0 \) to \( \mu > 0 \), \( \arg(\mu) \) changes by \( \pi \), and hence the orientation of the 3-cycle is reversed as \( \mu \) crosses 0.

To examine the stability of the 3-cycle for the map

\[ F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)z^2 + O(|z|^3), \]

we consider the map

\[ F_\mu^3(z) = (1 + 3\mu \lambda_1 + O(|\mu|^2))z + 3\lambda_0 c_{02}(0)z^2 + O(|\mu||z|^2 + |z|^3). \]

Then, we can easily see that the eigenvalues of the Jacobian \( \partial(F_\mu^3(z), F_\mu^3(z))/\partial(z, \bar{z}) \) are real and hyperbolic and hence the 3-cycle is hyperbolic (saddle) on both sides of \( \mu = 0 \).

Thus, we have the following conclusion:

**Theorem 1.** Let \( F_\mu : \mathbb{C} \rightarrow \mathbb{C} \) be the general map given in (2.7) and assume that the conditions (2.2) and (2.3) hold and that \( \lambda_0^3 = 1 \) (\( \lambda_0 \neq \pm 1 \)) and \( c_{02} \neq 0 \) and \( F_\mu(z) \) is put into the normal form (3.1).

Then a one-parameter family of 3-cycles \{\( (x_1(\mu), x_2(\mu), x_3(\mu)) \) | \( \mu \in \mathbb{R} \) \} undergoes transcritical bifurcation from the origin and they are
hyperbolic (saddle) on both sides of $\mu = 0$ and reverses the orientation as $\mu$ crosses 0. The 3-periodic points are given by (3.6)

(ii) The case $n = 3$ and $c_{02}(0) = 0$.

In this case, we can remove the second order term because the coefficient $\gamma_{02}(\mu)$ of the transformation $z = w + \psi(\mu, w, \bar{w})$, $\psi(\mu, w, \bar{w}) = \sum_{p+q \geq 2} \gamma_{pq}(\mu)w^p\bar{w}^q$ becomes regular for $\mu$ near 0. Hence, after the change of variables, $F_\mu$ takes the form

$$
(3.7) \quad F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2 + \beta(\mu)z^4 + \gamma(\mu)z^3 + O(|z|^5),
$$

where the coefficients $\alpha_0 \equiv \alpha(0), \beta_0 \equiv \beta(0)$ and $\gamma_0 \equiv \gamma(0)$ can be calculated from the original coefficients of $F_\mu(z)$, e.g.,

$$
\alpha_0 \equiv c_{21}(0) + \frac{|c_{11}(0)|^2}{1 - \lambda_0} + \frac{2\lambda_0 - 1}{\lambda_0(1 - \lambda_0)} c_{11}(0)c_{20}(0).
$$

Here we assume that $\alpha_0, \beta_0, \gamma_0 \not= 0$. From (2.9), we have

$$
(3.8) \quad \mu \lambda_1 z + \lambda_0 \alpha_0 z^2 \bar{z} + \bar{\lambda}_0 \beta_0 z^4 + \bar{\lambda}_0 \gamma_0 z\bar{z}^3
+ O(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0.
$$

Setting $z = re^{2\pi i\phi}$ and separating the trivial solution $r = 0$, we have

$$
(3.9) \quad 2\pi i\alpha + \bar{\lambda}_0 \alpha_0 r^2 + \bar{\lambda}_0 \beta_0 r^3 e^{6\pi i\phi} + \bar{\lambda}_0 \gamma_0 r^3 e^{-6\pi i\phi}
+ O(|\mu|^2 + |\mu|r^2 + |\mu|r^3 + r^4) = 0.
$$

Set

$$
(3.10) \quad \mu = \mu_0 r^2 + \mu_1 r^3, \quad \phi = \phi_0 + \phi_1,
$$

where $\mu_0, \mu_1, \phi_0$ and $\phi_1$ are to be determined. Substituting (3.10) into (3.9), we have

$$
(3.11) \quad (2\pi i\alpha_0 + \bar{\lambda}_0 \alpha_0)r^2
+ (2\pi i\alpha_1 + \bar{\lambda}_0 (\beta_0 e^{6\pi i\phi} + \gamma_0 e^{-6\pi i\phi}))r^3 + O(r^4) = 0.
$$
First, choose $\mu_0$ so that $2\pi i a \mu_0 + \bar{\lambda}_0 \alpha_0 = 0$. Then

\begin{equation}
(3.12) \quad \mu_0 = \begin{cases} 
\frac{|\alpha_0|}{2\pi a}, & \text{if } \arg \alpha_0 = \frac{\pi}{6} \ (\text{mod } 2\pi) \\
-\frac{|\alpha_0|}{2\pi a}, & \text{if } \arg \alpha_0 = -\frac{\pi}{6} \ (\text{mod } 2\pi).
\end{cases}
\end{equation}

Thus, note that if $\arg \alpha_0 \neq \pi/6 \ (\text{mod } \pi)$, there does not exist any 3-cycles bifurcating from the origin. Hence, from now on, we assume that

\begin{equation}
(3.13) \quad \arg \alpha_0 = \pi/6 \ (\text{mod } \pi).
\end{equation}

With this choice of $\mu_0$, (3.11) becomes

\begin{equation}
(3.14) \quad \bar{\lambda}_0 (\beta_0 e^{6\pi i \phi} + \gamma_0 e^{-6\pi i \phi}) + 2\pi i a \mu_1 + \mathcal{O}(r) = 0.
\end{equation}

In order to choose $\phi_0$ so that $\beta_0 e^{6\pi i \phi_0} + \gamma_0 e^{-6\pi i \phi_0} = 0$, we must have $|\beta_0| = |\gamma_0|$ and

\begin{equation}
(3.15) \quad \phi_0 = \frac{1}{12\pi} \arg \left( -\frac{\gamma_0}{\beta_0} \right) \ (\text{mod } 1/6).
\end{equation}

If $|\beta_0| \neq |\gamma_0|$, there is no 3-cycle bifurcating from the origin. So, here, we also assume that

\begin{equation}
(3.16) \quad |\beta_0| = |\gamma_0| \neq 0.
\end{equation}

Note that $\phi_0$ in (3.15) has two values

\begin{equation}
(3.17) \quad \phi_0^{(1)} = \frac{1}{12\pi} \arg \left( -\frac{\gamma_0}{\beta_0} \right) \ (\text{mod } 1/3),
\end{equation}

\begin{equation}
\phi_0^{(2)} = \frac{1}{12\pi} \arg \left( -\frac{\gamma_0}{\beta_0} \right) + 1/6 \ (\text{mod } 1/3).
\end{equation}

Now from (3.14), we let

\begin{equation}
h(\mu_1, r, \phi) = \bar{\lambda}_0 (\beta_0 e^{6\pi i \phi} + \gamma_0 e^{-6\pi i \phi}) + 2\pi i a \mu_1 + \mathcal{O}(r).
\end{equation}

Then by the implicit function theorem, we know that

\begin{equation}
\mu_1 = \mu_1^{(j)}(r) = \mathcal{O}(r), \phi = \phi^{(j)}(r) = \phi_0^{(j)} + \mathcal{O}(r) \ (j = 1, 2).
\end{equation}
Thus, we have a pair of 3-cycles $z = re^{2\pi i \phi(j)}(r) (j = 1, 2)$ on one side of $\mu = 0$, where $r$ is regarded as a parameter which is related to $\mu$ as

$$\begin{align*}
\mu^{(1)} &= \mu_0 r^2 + \mathcal{O}(r^4), \\
\phi^{(1)} &= \frac{1}{12\pi} \arg\left(-\frac{\gamma_0}{\beta_0}\right) + \mathcal{O}(r) \mod 1/3), \\
\mu^{(2)} &= \mu_0 r^2 - \mathcal{O}(r^4), \\
\phi^{(2)} &= \frac{1}{12\pi} \arg\left(-\frac{\gamma_0}{\beta_0}\right) + 1/6 + \mathcal{O}(r) \mod 1/3).
\end{align*}$$

(3.18)

Note that if $\arg \alpha_0 = \pi/6 \mod 2\pi$, we have a supercritical bifurcation and if $\arg \alpha_0 = 7\pi/6 \mod 2\pi$, subcritical bifurcation.

To study the stability of the pair of 3-cycles for the map

$$F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)z^4 + \gamma(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5),$$

we consider the map

$$F_\mu^3(z) = (1 + 6\pi i \lambda \mu)z + 3\lambda_0 \alpha_0 z^2\bar{z} + 3\lambda_0 \beta_0 z^4 + 3\lambda_0 \gamma_0 z\bar{z}^3$$

$$+ \mathcal{O}(|\mu|^2|z| + |\mu||z|^3 + |\mu||z|^4 + |z|^6).$$

Then we can easily check that one of the two 3-cycles on one side must be hyperbolic (saddle).

Therefore, we can state the following theorem.

**THEOREM 2.** Let $F_\mu : \mathbb{C} \to \mathbb{C}$ be the general map given in (2.7) and assume that the conditions (2.2) and (2.3) hold and that $\lambda_0^3 = 1(\lambda_0 \neq \pm 1)$, $c_{02}(0) = 0$ and $F_\mu(z)$ is put into a normal form (3.7).

Then, unless either $\arg \alpha_0 = \pi/6 \mod \pi$ or $|\beta_0| = |\gamma_0| (\neq 0)$, there is no bifurcation of 3-cycles from the origin.

If both conditions hold, then a pair of one-parameter family of 3-cycles $\{(x_1^{(r)}, x_2^{(r)}, x_3^{(r)}) \mid r \in \mathbb{R}^+, j = 1, 2\}$ undergoes a supercritical (if $\arg \alpha_0 = \pi/6 \mod 2\pi$) or subcritical (if $\arg \alpha_0 = 7\pi/6 \mod 2\pi$) bifurcation from the origin and the parameter $r$ is related to $\mu$ as in (3.18). Moreover, on either side of $\mu = 0$ one of two families of 3-cycles is hyperbolic (saddle).
4. Bifurcation analysis of 4-cycles

Let \( \lambda_0 = e^{2\pi i/4} = i \). The normal form of \( F_\mu(z) \) is

\[
F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2 + \beta(\mu)z^3 + O(|z|^5).
\]

where \( \alpha(0) = \alpha_0 \) and \( \beta(0) = \beta_0 \) can be computed from the coefficients of the original equation (see Kim ([12])) and we assume that \( \alpha_0, \beta_0 \neq 0 \). From (2.9), we have

\[
\mu_1 z + \bar{\lambda}_0 \alpha_0 z^2 + \bar{\lambda}_0 \beta_0 z^3 + g_1(\mu, z, \bar{z}) = 0,
\]

where \( g_1(\mu, z, \bar{z}) = O(|\mu|^2|z| + |\mu||z|^3 + |z|^5) \). Setting \( z = re^{2\pi i\phi} \) and separating the trivial solution \( r = 0 \), we have

\[
2\pi a_\mu - \alpha_0 r^2 - \beta_0 r^2 e^{-8\pi i\phi} + g(\mu, r, \phi) = 0,
\]

where \( g(\mu, r, \phi) = O(|\mu|^2 + |\mu| r^2 + r^4) \).

To look for the principal part, put

\[
\mu = \mu_0 r^2 + \mu_1 r^2, \quad \phi = \phi_0 + \phi_1,
\]

where \( \mu_0, \mu_1, \phi_0 \) and \( \phi_1 \) are to be determined. Substituting (4.4) in (4.3) and dividing by \( r^2 \), we have

\[
(2\pi a_\mu - \alpha_0 - \beta_0 e^{-8\pi i\phi_0}) + 2\pi a_\mu_1 + f_1(\mu, r, \phi) = 0,
\]

where \( f_1(\mu, r, \phi) = O(r^2), f_1(\mu, r, \phi + 1/4) = f_1(\mu, r, \phi) \). Choose \( \mu_0, \phi_0 \) so that

\[
2\pi a_\mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi_0} = 0.
\]

Then we must have

\[
|2\pi a_\mu_0 - \alpha_0| = |\beta_0|,
\]

\[
\phi_0 = -\frac{1}{8\pi} \arg \left( \frac{2\pi a_\mu_0 - \alpha_0}{\beta_0} \right) \pmod{1/4}.
\]

Let \( \mu_0^{(1)}, \mu_0^{(2)} \) be two solutions of \(|2\pi a_\mu_0 - \alpha_0| = |\beta_0|\). Then we have

\[
\mu_0^{(1,2)} = \frac{1}{2\pi a} \left\{ \text{Re} \alpha_0 \pm \sqrt{|\beta_0|^2 - |\text{Im} \alpha_0|^2} \right\}.
\]
Note that, since $\mu_0$ is real, we must have

$$(4.9) \quad |\text{Im } \alpha_0| \leq |\beta_0|. $$

That is, if $|\text{Im } \alpha_0| > |\beta_0|$, there does not exist any 4-cycles bifurcating from the origin. Assume that $(4.9)$ is satisfied. Once $\mu_0$ is determined from $(4.8)$, then we know from $(4.7)$ that $\phi_0$ has also two values

$$(4.10) \quad \phi_0^{(j)} = \frac{-1}{8\pi} \arg \left( \frac{2\pi a \mu_0^{(j)} - \alpha_0}{\beta_0} \right) \quad j = 1, 2. \quad (\text{mod } 1/4). $$

From $(4.5)$, let

$$h(\mu_1, r, \phi) = (2\pi a \mu_0 - \alpha_0 - \beta_0 e^{-8\pi i\phi}) + 2\pi a \mu_1 + f_1(\mu, r, \phi).$$

Note that in order to apply implicit function theorem, we must have the strictly in equality sign in $(4.9)$, i.e.,

$$(4.11) \quad |\text{Im } \alpha_0| < |\beta_0|. $$

Then under the assumption $(4.11)$, by the implicit function theorem, we know that

$$\mu_1 = \mu_1(r), \quad \phi = \phi(r), \quad \mu_1(0) = 0, \quad \phi(0) = \phi_0$$

since $f_1(\mu, r, \phi)$ is an even function of $r$ from the property in $(4.5)$. We also know that $\mu(r), \phi(r)$ are even functions of $r$. Thus, we have

$$(4.12) \quad \mu^{(j)}(r) = \mu_0^{(j)} r^2 + \mathcal{O}(r^4) \quad (j = 1, 2)$$

$$\phi^{(j)}(r) = \phi_0^{(j)} + \mathcal{O}(r^2)$$

where $\mu_0^{(1),(2)}, \phi_0^{(1),(2)}$ are given in $(4.8)$ and $(4.10)$. From $(4.8)$, we know that

$$\mu_0^{(1)} \cdot \mu_0^{(2)} = \frac{|\alpha_0|^2 - |\beta_0|^2}{4\pi^2 a^2}.$$ 

Hence we know that if $|\alpha_0| > |\beta_0|$, then $\mu_0^{(1)} \cdot \mu_0^{(2)} > 0$ and the so to families of 4-cycles bifurcate on the same side of $\mu = 0$. And if
Bifurcation of subharmonic orbits

149

\( |\alpha_0| < |\beta_0| \), then \( \mu_0^{(1)} \cdot \mu_0^{(2)} < 0 \) and so they bifurcate on the opposite sides of \( \mu = 0 \) (i.e., transcritical bifurcation). If \( |\alpha_0| = |\beta_0| \), then 

\[ \mu_0^{(1)} = 0, \mu_0^{(2)} = \frac{1}{\pi a} \Re \alpha_0 \neq 0 \]

and hence we know that

\[
\begin{align*}
\mu^{(1)}(r) &= \mathcal{O}(r^4), \\
\mu^{(2)}(r) &= \mu_0^{(2)} r^2 + \mathcal{O}(r^4).
\end{align*}
\]

(4.13)

To study the stability of the 4-cycles for the map

\[ F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + \mathcal{O}(|z|^5) \]

where \( \alpha(0) \equiv \alpha_0, \beta(0) \equiv \beta_0 \) are given by Kim ([12]), we consider the map

\[ F_\mu^4(z) = [1 + 8\pi i a \mu + \mathcal{O}(|\mu|^2)]z - 4i \alpha_0 z^2 \bar{z} - 4i \beta_0 \bar{z}^3 + \mathcal{O}(|\mu||z|^3 + |z|^5). \]

If \( \sigma_1, \sigma_2 \) are the eigenvalues of the linear part of \( F_\mu^4(z) \) at one of the 4 fixed points of one family, then we can easily see that if \( |\alpha_0| < |\beta_0| \), then \( \sigma_1, \sigma_2 \) are real hyperbolic and if \( |\alpha_0| > |\beta_0| \), then one of the families is hyperbolic. Hence we have the following conclusion:

**Theorem 3.** Let \( F_\mu : \mathbb{C} \rightarrow \mathbb{C} \) be the general map given in (2.7) and assume that the conditions (2.2) and (2.3) hold and that \( \lambda_0^4 = 1(\lambda_0 \neq \pm 1) \) and \( F_\mu(z) \) is put into the normal form (4.1), where \( \alpha_0 \neq 0, \beta_0 \neq 0 \).

Then if \( |\text{Im} \alpha_0| > |\beta_0| \), there is no bifurcation of 4-cycles from the origin. If \( |\text{Im} \alpha_0| < |\beta_0| \), then we have two one-parameter families of 4-cycles \( \{(x^{(j)}_1(r), x^{(j)}_2(r), x^{(j)}_3(r), x^{(j)}_4(r)) \mid r \in \mathbb{R}^+, j = 1, 2\} \) bifurcating from the origin and those 4-cycles are given by \( x^{(j)}_k(r) = x^{(j)}_k(r) = re^{2\pi i (\delta^{(j)} + \frac{k-1}{4})} + \mathcal{O}(r^3), (j = 1, 2, k = 1, 2, 3, 4) \) where the parameter \( r \) is related to \( \mu \) as in (4.4), (4.8) and (4.10).

Moreover, if \( |\alpha_0| > |\beta_0| \), then the two families bifurcate on the same sides of \( \mu = 0 \) and one of the families is hyperbolic (saddle) and if \( |\alpha_0| < |\beta_0| \), the two families bifurcate on the opposite side of \( \mu = 0 \) and both are hyperbolic.
References


Department of Mathematics
University of Ulsan
Ulsan, Korea 680–749