FUZZY Γ-RINGS

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The concept of a fuzzy set, introduced by Zadeh ([6]), was applied in [2] to generalize some of the basic concepts of general topology. Rosenfeld ([5]) applied this concept to the theory of groupoids and groups. The present paper constitutes a similar application to the elementary theory of Γ-rings.

We recall that a fuzzy set in a set \( S \) is a function \( \mu \) from \( S \) into \([0, 1]\). Let \( \mu \) and \( \nu \) be fuzzy sets in a set \( S \). Then we define

\[
\mu = \nu \iff \mu(x) = \nu(x) \quad \text{for all } x \in S.
\]

\[
\mu \subseteq \nu \iff \mu(x) \leq \nu(x) \quad \text{for all } x \in S.
\]

\[
(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\} \quad \text{for all } x \in S.
\]

\[
(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \text{for all } x \in S.
\]

More generally, for a family of fuzzy sets, \( \{\mu_i \mid i \in I\} \), we define

\[
(\bigcup_{i \in I} \mu_i)(x) = \sup_{i \in I} \{\mu_i(x)\}, \quad x \in S
\]

\[
(\bigcap_{i \in I} \mu_i)(x) = \inf_{i \in I} \{\mu_i(x)\}, \quad x \in S.
\]

**Definition 1.** ([1]) If \( M = \{x, y, z, \ldots\} \) and \( \Gamma = \{\alpha, \beta, \gamma, \ldots\} \) are additive abelian groups, and for all \( x, y, z \) in \( M \) and all \( \alpha, \beta \) in \( \Gamma \), the following conditions are satisfied

1. \( x\alpha y \) is an element of \( M \),
2. \( (x + y)\alpha z = x\alpha z + y\alpha z \), \( x(\alpha + \beta)y = x\alpha y + x\beta y \), \( x\alpha(y + z) = x\alpha y + x\alpha z \),
3. \( (x\alpha y)\beta z = x\alpha(y\beta z) \),

then \( M \) is called a Γ-ring.

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DEFINITION 2. ([1]) A subset $A$ of the $\Gamma$-ring $M$ is a left (right) ideal of $M$ if $A$ is an additive subgroup of $M$ and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}(\Gamma M)$$

is contained in $A$. If $A$ is both a left and a right ideal, then $A$ is a two-sided ideal, or simply an ideal of $M$.

DEFINITION 3. A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is called a fuzzy left (right) ideal of $M$ if

- $(4) \mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
- $(5) \mu(x\alpha y) \geq \mu(y) \quad (\mu(x\alpha y) \geq \mu(x))$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set $\mu$ in a $\Gamma$-ring $M$ is called a fuzzy ideal of $M$ if $\mu$ is both a fuzzy left and a fuzzy right ideal of $M$.

We note that $\mu$ is a fuzzy ideal of $M$ if and only if

- $(4) \mu(x - y) \geq \min\{\mu(x), \mu(y)\}$,
- $(6) \mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

Throughout this paper, all proofs are going to proceed the only left cases, because the right cases are obtained from similar method. We denote $0_M$ the zero element of a $\Gamma$-ring $M$.

PROPOSITION 1. If $\mu$ is a fuzzy left (right) ideal of a $\Gamma$-ring $M$, then

- $(7) \mu(0_M) \geq \mu(x)$,
- $(8) \mu(-x) = \mu(x)$,
- $(9) \mu(x - y) = \mu(0_M)$ implies $\mu(x) = \mu(y)$,

for all $x, y \in M$.

Proof. (7) We have that for any $x \in M$,

$$\mu(0_M) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

(8) By (7), we have that

$$\mu(-x) = \mu(0_M - x) \geq \min\{\mu(0_M), \mu(x)\} = \mu(x)$$

for all $x \in M$. Since $x$ is arbitrary, we conclude that $\mu(-x) = \mu(x)$. 
(9) Assume that \( \mu(x - y) = \mu(0_M) \) for all \( x, y \in M \). Then
\[
\mu(x) = \mu(x - y + y) \\
\geq \min\{\mu(x - y), \mu(y)\} \\
= \min\{\mu(0_M), \mu(y)\} \\
= \mu(y).
\]
Similarly, using \( \mu(y - x) = \mu(x - y) = 0 \), we have \( \mu(y) \geq \mu(x) \).

**Example 1.** If \( G \) and \( H \) are additive abelian groups and \( M = \text{Hom}(G, H), \Gamma = \text{Hom}(H, G) \) then \( M \) is a \( \Gamma \)-ring with the operations pointwise addition and composition of homomorphisms ([1]). Define a fuzzy set \( \mu : M \rightarrow [0, 1] \) by \( \mu(0_M) = t_1, \mu(f) = t_2, 0 \leq t_2 < t_1 \leq 1 \), where \( f \) is any member of \( M \) with \( f \neq 0_M \). Routine calculations give that \( \mu \) is a fuzzy left (right) ideal of \( M \).

**Theorem 1.** If \( \mu \) is a fuzzy left (right) ideal of a \( \Gamma \)-ring \( M \), then the set
\[
A := \{ x \in M | \mu(x) = \mu(0_M) \}
\]
is a left (right) ideal of \( M \).

**Proof.** Let \( x, y \in A \). Then by (4),
\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = \mu(0_M).
\]
It follows from (7) that \( \mu(x - y) = \mu(0_M) \), so that \( x - y \in A \). This means that \( A \) is an additive subgroup of \( M \). Now let \( u \in A, \alpha \in \Gamma \) and \( x \in M \). Then by (5), \( \mu(x\alpha u) \geq \mu(u) = \mu(0_M) \) and so \( \mu(x\alpha u) = \mu(0_M) \). Therefore \( x\alpha u \in A \). This completes the proof.

**Theorem 2.** The intersection of any family of fuzzy left (right) ideals of a \( \Gamma \)-ring \( M \) is also a fuzzy left (right) ideal of \( M \).

**Proof.** Let \( \{\mu_i\} \) be a family of fuzzy left ideals of a \( \Gamma \)-ring \( M \). Then for every \( x, y \in M \) and \( \alpha \in \Gamma \),
\[
(\cap \mu_i)(x - y) = \inf\{\mu_i(x - y)\} \\
\geq \inf\{\min\{\mu_i(x), \mu_i(y)\}\} \\
= \min\{\inf \mu_i(x), \inf \mu_i(y)\} \\
= \min\{(\cap \mu_i)(x), (\cap \mu_i)(y)\}
\]
and
\[(\cap \mu_t)(x \alpha y) = \inf \{\mu_t(x \alpha y)\} \geq \inf \{\mu_t(y)\} = (\cap \mu_t)(y)\].

**Definition 4.** ([3]) Let \(\mu\) be a fuzzy set in a set \(S\). For \(t \in [0,1]\), the set
\[\mu_t := \{x \in S | \mu(x) \geq t\}\]
is called a level subset of \(\mu\).

**Theorem 3.** Let \(\mu\) be a fuzzy set in a \(\Gamma\)-ring \(M\). Then
(a) if \(\mu\) is a fuzzy left (right) ideal of \(M\), then \(\mu_t\) is a left (right) ideal of \(M\) for all \(t \in [0,\mu(0_M)]\) which is called the level left (right) ideal of \(M\).
(b) if \(\mu_t\) is a left (right) ideal of \(M\) for all \(t \in Im(\mu)\), then \(\mu\) is a fuzzy left (right) ideal of \(M\).

*Proof.* (a) Assume that \(\mu\) is a fuzzy left ideal of \(M\). Let \(x, y \in \mu_t\).
Then \(\mu(x) \geq t\) and \(\mu(y) \geq t\). It follows that
\[\mu(x - y) \geq \min \{\mu(x), \mu(y)\} \geq t,\]
and that \(x - y \in \mu_t\). Now let \(x \in M, \alpha \in \Gamma\) and \(y \in \mu_t\). Since \(\mu\) is a fuzzy left ideal, \(\mu(x \alpha y) \geq \mu(y) \geq t\). Thus \(x \alpha y \in \mu_t\). Therefore \(\mu_t\) is a left ideal of \(M\).
(b) Let \(\mu_t\) be a left ideal of \(M\). We must prove that (4) and (5) hold. If (4) is not true, then
\[\mu(x - y) < \min \{\mu(x), \mu(y)\}\]
for some \(x, y \in M\). For these elements \(x, y\), there exist \(t_i, t_j \in Im(\mu)\), say \(t_i < t_j\), such that \(\mu(x) = t_i, \mu(y) = t_j\). Then
\[\mu(x - y) < \min \{\mu(x), \mu(y)\} = t_i,\]
and so \(x - y \notin \mu_t\). This is a contradiction. If (5) is not true, then for a fixed \(\alpha \in \Gamma\), there exist \(x, y \in M\) such that \(\mu(x \alpha y) < \mu(y)\). Let \(s_i, s_j \in Im(\mu)\) be such that \(s_i < s_j\), \(\mu(x) = s_i\) and \(\mu(y) = s_j\). Then \(\mu(x \alpha y) < \mu(y) = s_j\) and so \(x \alpha y \notin \mu_{s_j}\), a contradiction. This completes the proof.
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**THEOREM 4.** Let \( A \) be a left (right) ideal of a Γ-ring \( M \). Then for any \( t \in (0,1) \), there exists a fuzzy left (right) ideal \( \mu \) of \( M \) such that \( \mu_t = A \).

**Proof.** Let \( \mu : M \to [0,1] \) be a fuzzy set defined by

\[
\mu(x) = \begin{cases} 
  t & \text{if } x \in A, \\
  0 & \text{if } x \not\in A,
\end{cases}
\]

where \( t \) is a fixed number in \((0,1)\). Then clearly \( \mu_t = A \). Let \( x, y \in M \) and \( \alpha \in \Gamma \). By routine calculations, we have that

\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}.
\]

Now if \( y \in A \), then \( xay \in A \) because \( A \) is a left ideal of \( M \). Hence \( \mu(xay) = t = \mu(y) \). If \( y \not\in A \), then \( \mu(y) = 0 \) and so \( \mu(xay) \geq \mu(y) \).

Therefore \( \mu \) is a fuzzy left ideal of \( M \).

**THEOREM 5.** Let \( \mu \) be a fuzzy left (right) ideal of a Γ-ring \( M \). Then two level left (right) ideals \( \mu_{t_1} \) and \( \mu_{t_2} \) (with \( t_1 < t_2 \)) of \( \mu \) are equal if and only if there is no \( x \in M \) such that \( t_1 \leq \mu(x) < t_2 \).

**Proof.** (\( \Rightarrow \)) Suppose \( t_1 < t_2 \) and \( \mu_{t_1} = \mu_{t_2} \). If there exists \( x \in M \) such that \( t_1 \leq \mu(x) < t_2 \), then \( \mu_{t_2} \) is a proper subset of \( \mu_{t_1} \). This is a contradiction.

(\( \Leftarrow \)) Assume that there is no \( x \in M \) such that \( t_1 \leq \mu(x) < t_2 \). From \( t_1 < t_2 \) it follows that \( \mu_{t_2} \subseteq \mu_{t_1} \). If \( x \in \mu_{t_1} \), then \( \mu(x) \geq t_1 \) and so \( \mu(x) \geq t_2 \) because \( \mu(x) \not\in t_2 \). Hence \( x \in \mu_{t_2} \). This completes the proof.

**THEOREM 6.** Let \( M \) be a Γ-ring and \( \mu \) a fuzzy left (right) ideal of \( M \). If \( \text{Im}(\mu) = \{t_1, ..., t_n\} \), where \( t_1 < ... < t_n \), then the family of left (right) ideals \( \mu_{t_i} (i = 1, ..., n) \) constitutes all the level left (right) ideals of \( \mu \).

**Proof.** Let \( t \in [0,1] \) and \( t \not\in \text{Im}(\mu) \). If \( t < t_1 \), then \( \mu_{t_1} \subseteq \mu_t \). Since \( \mu_{t_1} = M \), it follows that \( \mu_t = M \), so that \( \mu_t = \mu_{t_1} \). If \( t_i < t < t_{i+1} (1 \leq i \leq n - 1) \) then there is no \( x \in M \) such that \( t \leq \mu(x) < t_{i+1} \). From Theorem 5, we have that \( \mu_t = \mu_{t_{i+1}} \). This shows that for any \( t \in [0,1] \) with \( t \leq \mu(0_M) \), the level left ideal \( \mu_t \) is in \( \{\mu_i | 1 \leq i \leq n\} \).
THEOREM 7. Let $A$ be a nonempty subset of a $\Gamma$-ring $M$ and let
\( \mu \) be a fuzzy set in $M$ such that $\mu$ is into \{0,1\}, so that $\mu$ is the characteristic function of $A$. Then $\mu$ is a fuzzy left (right) ideal of $M$ if and only if $A$ is a left (right) ideal of $M$.

Proof. Assume that $\mu$ is a fuzzy left ideal of $M$. Let $x, y \in A$. Then $\mu(x) = \mu(y) = 1$. Thus $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = 1$ and so $\mu(x - y) = 1$. This means that $x - y \in A$. Therefore $A$ is an additive subgroup of $M$. Let $x \in M, y \in A$ and $\alpha \in \Gamma$. Then $\mu(x\alpha y) \geq \mu(y) = 1$ and hence $\mu(x\alpha y) = 1$. So $x\alpha y \in A$, and $A$ is a left ideal of $M$. The proof of converse is similar to that of Theorem 4.

DEFINITION 5. ([1]) Let $M$ and $N$ both be $\Gamma$-rings, and $\theta$ a mapping of $M$ into $N$. Then $\theta$ is a $\Gamma$-homomorphism iff $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

DEFINITION 6. ([5]) If $\mu$ is a fuzzy set in $M$, and $f$ is a function defined on $M$, then the fuzzy set $\nu$ in $f(M)$ defined by

\[ \nu(y) = \sup_{x \in f^{-1}(y)} \mu(x) \]

for all $y \in f(M)$ is called the image of $\mu$ under $f$. Similarly, if $\nu$ is a fuzzy set in $f(M)$, then the fuzzy set $\mu = \nu \circ f$ in $M$ (that is, the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in M$) is called the preimage of $\nu$ under $f$.

THEOREM 8. A $\Gamma$-homomorphic preimage of a fuzzy left (right) ideal is a fuzzy left (right) ideal.

Proof. Let $\theta : M \rightarrow N$ be a $\Gamma$-homomorphism of $\Gamma$-rings, $\nu$ a fuzzy left ideal of $N$ and $\mu$ the preimage of $\nu$ under $\theta$. Then

\[
\mu(x - y) = \nu(\theta(x - y))
\]

\[
= \nu(\theta(x) - \theta(y))
\]

\[
\geq \min\{\nu(\theta(x)), \nu(\theta(y))\}
\]

\[
= \min\{\mu(x), \mu(y)\}
\]
and
\[ \mu(x\alpha y) = \nu(\theta(x\alpha y)) \]
\[ = \nu(\theta(x)\alpha \theta(y)) \]
\[ \geq \nu(\theta(y)) \]
\[ = \mu(y) \]
for all \( x, y \in M \) and \( \alpha \in \Gamma \).

We say that a fuzzy set \( \mu \) in \( M \) has the sup property ([5]) if, for any subset \( T \) of \( M \), there exists \( t_0 \in T \) such that
\[ \mu(t_0) = \sup_{t \in T} \mu(t) \]

**Theorem 9.** A \( \Gamma \)-homomorphic image of a fuzzy left (right) ideal which has the sup property is a fuzzy left (right) ideal.

**Proof.** Let \( \theta : M \to N \) be a \( \Gamma \)-homomorphism of \( \Gamma \)-rings, \( \mu \) a fuzzy left ideal of \( M \) with the sup property and \( \nu \) the image of \( \mu \) under \( \theta \). Given \( \theta(x), \theta(y) \in \theta(M) \), let \( x_0 \in \theta^{-1}(\theta(x)) \), \( y_0 \in \theta^{-1}(\theta(y)) \) be such that
\[ \mu(x_0) = \sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \quad \mu(y_0) = \sup_{t \in \theta^{-1}(\theta(y))} \mu(t) \]
respectively. Then
\[ \nu(\theta(x) - \theta(y)) = \sup_{z \in \theta^{-1}(\theta(x) - \theta(y))} \mu(z) \]
\[ \geq \mu(x_0 - y_0) \]
\[ \geq \min\{\mu(x_0), \mu(y_0)\} \]
\[ = \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \sup_{t \in \theta^{-1}(\theta(y))} \mu(t)\} \]
\[ = \min\{\nu(\theta(x)), \nu(\theta(y))\} \]
and for any \( \alpha \in \Gamma \),
\[ \nu(\theta(x)\alpha \theta(y)) = \sup_{z \in \theta^{-1}(\theta(x)\alpha \theta(y))} \mu(z) \]
\[ \geq \mu(x_0 \alpha y_0) \]
\[ \geq \mu(y_0) \]
\[ = \sup_{t \in \theta^{-1}(\theta(y))} \mu(t) \]
\[ = \nu(\theta(y)) \].
This completes the proof.

References


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