FUZZY RELATIONS ON BCK/BCI—ALGEBRAS

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1. Introduction and preliminaries

The notion of a fuzzy subset and a fuzzy relation on a set was introduced by Zadeh ([6],[7]). Fuzzy relations on a group have been studied by Bhattacharya and Mukherjee ([1]). The concept of a fuzzy subalgebra of a BCK-algebra was introduced by Xi ([5]). In [3] the second author together with J. Meng solved the problem of classifying fuzzy subalgebras by their family of level subalgebras in BCK/BCI-algebras. In this paper we study fuzzy relations on BCK/BCI-algebras. We prove the following results. (i) If \( \mu \) and \( \sigma \) are fuzzy subalgebras of a BCK/BCI-algebra \( X \), then \( \mu \times \sigma \) is a fuzzy subalgebra of \( X \times X \). (ii) If \( \mu \times \sigma \) is a fuzzy subalgebra of \( X \times X \), then either \( \mu \) or \( \sigma \) is a fuzzy subalgebra of \( X \). (iii) If \( \sigma \) is a fuzzy subset of a BCK/BCI-algebra \( X \) and \( \mu_\sigma \) is the strongest fuzzy relation on \( X \) that is a fuzzy relation on \( \sigma \), then \( \mu_\sigma \) is a fuzzy subalgebra if and only if \( \sigma \) is a fuzzy subalgebra. An example is given to show that if \( \mu \times \sigma \) is a fuzzy subalgebra of \( X \times X \), then \( \mu \) and \( \sigma \) both need not be fuzzy subalgebras of \( X \).

We recall some definitions and results.

**Definition 1.1.** A fuzzy subset of any set \( S \) is a function \( \mu : S \to [0, 1] \).

**Definition 1.2.** ([2]) Let \( \mu \) be a fuzzy subset of a set \( S \). For \( t \in [0, 1] \), the set

\[
\mu_t := \{ x \in S | \mu(x) \geq t \}
\]

is called a level subset of \( \mu \).

**Definition 1.3.** ([3],[5]) Let \( X \) be a BCK/BCI-algebra. A fuzzy subset \( \mu \) of \( X \) is called a fuzzy subalgebra of \( X \) if for all \( x, y \in X \),

\[
\mu(x * y) \geq \min(\mu(x), \mu(y)).
\]

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**Lemma 1.4.** ([3]) Let \( X \) be a BCK/BCI-algebra and let \( \mu \) be a fuzzy subset of \( X \) such that \( \mu_t \) is a subalgebra of \( X \) for all \( t \in [0,1] \). Then \( \mu \) is a fuzzy subalgebra of \( X \).

**Definition 1.5.** ([3]) Let \( X \) be a BCK/BCI-algebra and let \( \mu \) be a fuzzy subalgebra of \( X \). The subalgebras \( \mu_t, t \in [0,1] \) and \( t \leq \mu(0) \), are called level subalgebras of \( \mu \).

**Definition 1.6.** ([1]) Let \( S \) be any set. A fuzzy relation \( \mu \) on \( S \) is a fuzzy subset of \( S \times S \), that is, a map \( \mu : S \times S \to [0,1] \).

**Definition 1.7.** ([1]) If \( \mu \) is a fuzzy relation on a set \( S \) and \( \sigma \) is a fuzzy subset of \( S \), then \( \mu \) is a fuzzy relation on \( \sigma \) if

\[ \mu(x,y) \leq \min(\sigma(x),\sigma(y)) \]

for all \( x, y \in S \).

**Definition 1.8.** ([1]) Let \( \mu \) and \( \sigma \) be fuzzy subsets of a set \( S \). The Cartesian product of \( \mu \) and \( \sigma \) is defined by

\[ (\mu \times \sigma)(x,y) = \min(\mu(x),\sigma(y)) \]

for all \( x, y \in S \).

**Lemma 1.9.** ([1]) Let \( \mu \) and \( \sigma \) be fuzzy subsets of a set \( S \). Then

(i) \( \mu \times \sigma \) is a fuzzy relation on \( S \),

(ii) \( (\mu \times \sigma)_t = \mu_t \times \sigma_t \) for all \( t \in [0,1] \).

**Definition 1.10.** ([1]) If \( \sigma \) is a fuzzy subset of a set \( S \), the strongest fuzzy relation on \( S \) that is a fuzzy relation on \( \sigma \) is \( \mu_\sigma \), given by

\[ \mu_\sigma(x,y) = \min(\sigma(x),\sigma(y)) \]

for all \( x, y \in S \).

**Lemma 1.11.** ([1]) For a given fuzzy subset \( \sigma \) of a set \( S \), let \( \mu_\sigma \) be the strongest fuzzy relation on \( S \). Then for \( t \in [0,1] \), we have that

\[ (\mu_\sigma)_t = \sigma_t \times \sigma_t. \]
2. Fuzzy relations on BCK/BCI-algebras

Lemma 2.1. ([5]) If \( \mu \) is any fuzzy subalgebra of a BCK/BCI-algebra \( X \), then \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

Proposition 2.2. Let \( \mu \) be a fuzzy subalgebra of a BCK/BCI-algebra \( X \) and let \( x \in X \). If \( \mu(x \ast y) = \mu(y) \) for every \( y \in X \), then \( \mu(x) = \mu(0) \).

Proof. For a fixed element \( x \in X \), suppose that \( \mu(x \ast y) = \mu(y) \) for every \( y \in X \). Choosing \( y = 0 \); then we have that \( \mu(x) = \mu(x \ast 0) = \mu(0) \).

Proposition 2.3. For a given fuzzy subset \( \sigma \) of a BCK/BCI-algebra \( X \), let \( \mu_\sigma \) be the strongest fuzzy relation on \( X \). If \( \mu_\sigma \) is a fuzzy subalgebra of \( X \times X \), then \( \sigma(x) \leq \sigma(0) \) for all \( x \in X \).

Proof. From the fact that \( \mu_\sigma \) is a fuzzy subalgebra of \( X \times X \), it follows from Lemma 2.1 that for every \( x \in X \),

\[
\mu_\sigma(x, x) \leq \mu_\sigma(0, 0),
\]

where \( (0, 0) \) is the zero element of \( X \times X \). But (*) means that

\[
\min(\sigma(x), \sigma(x)) \leq \min(\sigma(0), \sigma(0)),
\]

which implies that \( \sigma(x) \leq \sigma(0) \).

The following proposition is an immediate consequence of Lemma 1.11, and we omit the proof.

Proposition 2.4. If \( \sigma \) is a fuzzy subalgebra of a BCK/BCI-algebra \( X \), then the level subalgebras of \( \mu_\sigma \) are given by \( (\mu_\sigma)_t = \sigma_t \times \sigma_t \) for all \( t \in [0, 1] \).

Theorem 2.5. Let \( \mu \) and \( \sigma \) be fuzzy subalgebras of a BCK/BCI-algebra \( X \). Then \( \mu \times \sigma \) is a fuzzy subalgebra of \( X \times X \).

Proof. For any \( (x, y), (u, v) \in X \times X \), we have that

\[
(\mu \times \sigma)((x, y) \ast (u, v))
= (\mu \times \sigma)(x \ast u, y \ast v)
\geq \min(\mu(x \ast u), \sigma(y \ast v))
\geq \min(\min(\mu(x), \mu(u)), \min(\sigma(y), \sigma(v)))
\geq \min(\min(\mu(x), \sigma(y)), \min(\mu(u), \sigma(v)))
= \min((\mu \times \sigma)(x, y), (\mu \times \sigma)(u, v)).
\]
This completes the proof.

**Theorem 2.6.** Let $\mu$ and $\sigma$ be fuzzy subsets of a BCK/BCI-algebra $X$ such that $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$. Then either $\mu$ or $\sigma$ is a fuzzy subalgebra of $X$.

**Proof.** Assume that $\mu$ and $\sigma$ both are not fuzzy subalgebras of $X$. Then

$$\mu(x \ast y) < \min(\mu(x), \mu(y)) \quad \text{and} \quad \sigma(u \ast v) < \min(\sigma(u), \sigma(v))$$

for some $x, y, u, v \in X$. Now

$$\begin{align*}
(\mu \times \sigma)((x, u) \ast (y, v)) &= (\mu \times \sigma)(x \ast y, u \ast v) \\
&= \min(\mu(x \ast y), \sigma(u \ast v)) \\
&< \min(\min(\mu(x), \mu(y)), \min(\sigma(u), \sigma(v)))) \\
&= \min(\min(\mu(x), \sigma(u)), \min(\mu(y), \sigma(v)))) \\
&= \min((\mu \times \sigma)(x, u), (\mu \times \sigma)(y, v)),
\end{align*}$$

which is a contradiction. This completes the proof.

Now we give an example to show that if $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$, then $\mu$ and $\sigma$ both need not be fuzzy subalgebras of $X$.

**Example.** Let $X$ be a nonzero BCK/BCI-algebra and let $t, s \in [0, 1]$ be such that $0 \leq s \leq t < 1$. Define fuzzy subsets $\mu, \sigma : X \to [0, 1]$ by $\mu(x) = s$ and

$$\sigma(x) = \begin{cases} 
1 & \text{if } x = 0, \\
1 & \text{if } x \neq 0,
\end{cases}$$

for all $x \in X$, respectively. Then $(\mu \times \sigma)(x, y) = \min(\mu(x), \sigma(y)) = s$ for all $(x, y) \in X \times X$, that is, $\mu \times \sigma : X \times X \to [0, 1]$ is a constant function. Hence $\mu \times \sigma$ is a fuzzy subalgebra of $X \times X$. Now $\mu$ is a fuzzy subalgebra of $X$, but $\sigma$ is not a fuzzy subalgebra of $X$ since for $x \neq 0$ we have $\sigma(x \ast x) = \sigma(0) = t < 1 = \min(\sigma(x), \sigma(x))$. 

THEOREM 2.7. Let $\sigma$ be a fuzzy subset of a BCK/BCI-algebra $X$. Then $\sigma$ is a fuzzy subalgebra of $X$ if and only if $\mu_\sigma$ is a fuzzy subalgebra of $X \times X$.

Proof. ($\Rightarrow$) Assume that $\sigma$ is a fuzzy subalgebra of $X$. We claim that for any $(x_1, x_2), (y_1, y_2) \in X \times X$,

$$\mu_\sigma((x_1, x_2) \ast (y_1, y_2)) \geq \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)).$$

Since $\sigma$ is a fuzzy subalgebra, we have that

$$\sigma(x_1 \ast y_1) \geq \min(\sigma(x_1), \sigma(y_1))$$

and

$$\sigma(x_2 \ast y_2) \geq \min(\sigma(x_2), \sigma(y_2)).$$

Hence

$$\mu_\sigma((x_1, x_2) \ast (y_1, y_2))$$

$$= \mu_\sigma(x_1 \ast y_1, x_2 \ast y_2)$$

$$= \min(\sigma(x_1 \ast y_1), \sigma(x_2 \ast y_2))$$

$$\geq \min(\min(\sigma(x_1), \sigma(y_1)), \min(\sigma(x_2), \sigma(y_2)))$$

$$= \min(\min(\sigma(x_1), \sigma(y_2))), \min(\sigma(y_1), \sigma(y_2)))$$

$$= \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)),$$

and so the necessity is completed.

(\Leftarrow) Suppose that $\mu_\sigma$ is a fuzzy subalgebra of $X \times X$. Let $x_i, y_i \in X; i = 1, 2$. Then

$$\mu_\sigma(x_1 \ast y_1, x_2 \ast y_2) = \mu_\sigma((x_1, x_2) \ast (y_1, y_2))$$

$$\geq \min(\mu_\sigma(x_1, x_2), \mu_\sigma(y_1, y_2)).$$

This means that

$$\min(\sigma(x_1 \ast y_1), \sigma(x_2 \ast y_2)) \geq \min(\min(\sigma(x_1), \sigma(x_2)), \min(\sigma(y_1), \sigma(y_2))),$$

which implies that

$$\sigma(x_1 \ast y_1) \geq \min(\min(\sigma(x_1), \sigma(x_2)), \min(\sigma(y_1), \sigma(y_2))).$$

In particular, if we take $x_2 = 0 = y_2$, then by Proposition 2.3,

$$\sigma(x_1 \ast y_1) \geq \min(\min(\sigma(x_1), \sigma(0)), \min(\sigma(y_1), \sigma(0)))$$

$$= \min(\sigma(x_1), \sigma(y_1)).$$

Hence $\sigma$ is a fuzzy subalgebra of $X$. 
References


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