ON SEVERAL CONTINUOUS FUNCTIONS
ON FUZZY CONVERGENCE SPACES

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1. Introduction

The convergence function between the filters on a given set \( S \) and the subsets of \( S \) was introduced by D.C. Kent ([12]) in 1964 and it may be regarded as a generation of a topological space and further studied by many authors.

After Zadech created fuzzy sets in his classical paper ([14]), Chang ([5]) used them to introduce the concept of a fuzzy sets using metric defined as the Hausdorff metric between the supported endographs. Recently, B.Y. Lee and J.H. Park ([16]) defined a new structure, called by fuzzy convergence structure, using prefilter.

In this paper, we define new the several continuous functions between the fuzzy convergence spaces, that is, fuzzy super continuity, fuzzy \( \delta \)-continuity, and fuzzy weakly \( \delta \)-continuity, and introduce the relationships between them. And we introduce the concepts of initial fuzzy convergence structures and product fuzzy convergence spaces and investigate their properties.

2. Preliminaries

The reader is asked to refer to [14], [5], [19], [21] and [22], for fuzzy sets and fuzzy topological spaces.

Let \( X \) be a nonempty set and \( I \) the unit closed interval \( I=\{0,1\} \). A fuzzy set \( A \) in \( X \) is an element of the set \( F(X) \) of all functions from \( X \) into \( I \) and the elements of \( F(X) \) are called fuzzy subsets ([14]). For fuzzy set \( A \) and \( B \) in \( X \), \( A \subseteq B \) if \( A(x) \leq B(x) \) for all \( x \) in \( X \). The symbol \( \phi \) is used to denote the empty fuzzy set \( \phi(x) = 0 \) for all \( x \in X \) and for \( X \) we have the definition \( X(x) = 1 \) for all \( x \in X \). A fuzzy
point \( p \) in \( X \) is fuzzy set in \( X \) defined by \( p(x) = \lambda \) (0 < \( \lambda \) ≤ 1) for \( x = x_p \) and \( p(x) = 0 \) for \( x \neq x_p \). Then, we call \( x_p \) the support of \( p \) and \( \lambda \) the value of \( p \). A fuzzy point \( p \in A \), where \( A \) is a fuzzy set in \( X \), if \( p(x_p) \leq A(x_p) \).

A fuzzy point \( p \) is said to be quasi coincident with \( A \), denoted by \( pqA \), if \( p(x_p) + A(x_p) > 1 \) for a fuzzy point \( p \) and a fuzzy set \( A \) (see in [21]). A fuzzy set \( A \) is said to be quasi coincident with a fuzzy set \( B \), denoted by \( AqB \), if there exists some \( x \) in \( X \) such that \( A(x) + B(x) > 1 \).

A fuzzy point \( \eta \) is said to be quasi coincident with \( A \), denoted by \( \eta_{\alpha}A \) if there exists some \( x \) in \( X \) such that \( A(x) + \eta_{\alpha}(x) > 1 \).

\( \eta_{\alpha}A \) is called a \( \eta_{\alpha} \)-neighborhood of \( p \) (in short \( \eta_{\alpha}-nbd \)) if there exists some \( B \) in \( \tau X \) so that \( B \leq A \) and \( pqB \).

A function \( f \) is said to be fuzzy continuous if \( \tau X \) whenever \( B \in \tau Y \), where \( \tau X \) and \( \tau Y \) are the fuzzy topologies on \( X \) and \( Y \), respectively.

3. Fuzzy convergence spaces

In this section, we introduce fuzzy convergence spaces using prefilters, and we define the set functions \( \Gamma_C, I_C \) and introduce their properties.

DEFINITION 3.1. ([4]) A prefilter on \( X \) is a nonempty subset \( \Phi \) of the set \( I^X \) of functions from \( X \) into closed interval \( I = [0, 1] \) with the properties:
(1) If \( A, B \in \Phi \), then \( A \cap B \in \Phi \).
(2) If \( A \in \Phi \) and \( A \subseteq B \), then \( B \in \Phi \).
(3) \( \Phi \notin \Phi \).

If \( \Phi \) and \( \Psi \) are prefilters on \( X \), \( \Phi \) is said to be finer than \( \Psi \) (\( \Psi \) is coarser than \( \Phi \)) if and only if \( \Psi \subseteq \Phi \). A prefilter \( \Phi \) on \( X \) is said to be ultra prefilter if it is no other prefilter finer than \( \Phi \) (i.e., it is maximal for the inclusion relation among prefilters).

A prefilterbase on \( X \) is the nonempty subset \( \beta \) of \( I^X \) with the properties:

(1) If \( A, B \in \beta \), there exists \( C \in \beta \) such that \( C \subseteq A \cap B \).
(2) \( \Phi \notin \beta \).

If \( \beta \) is a prefilterbase then \( \langle \beta \rangle = \{ A \in I^X : B \subseteq A \) for some \( B \in \beta \} \) is a prefilter. If \( \langle \beta \rangle = \Phi \), we say that \( \beta \) is a prefilterbase for the prefilter \( \Phi \), or that \( \beta \) generates \( \Phi \).

We define convergence structure by prefilter, called fuzzy convergence structure. For nonempty universal set \( X \), \( P(X) \) denotes the set of all prefilters on \( X \) and \( F(X) \) the set of all fuzzy sets on \( X \). For each fuzzy point \( p \) in \( X \), \( \hat{p} \) is denoted by

\[
\{ A \in I^X : pqA \}
\]

Let \( f \) be a function from \( X \) into \( Y \). Then for a fuzzy point \( p \) in fuzzy set \( A \) in \( X \), \( f(p) \in f(A) \) and for two prefilters \( \mathcal{F}, \mathcal{G} \) on \( X \), \( f(\mathcal{F} \cap \mathcal{G}) = f(\mathcal{F}) \cap f(\mathcal{G}) \) and so \( f(\mathcal{F} \cap \hat{p}) = f(\mathcal{F}) \cap f(\hat{p}) \) and \( f(\hat{p}) = f(\hat{p}) \). For a fuzzy prefilter \( \mathcal{F} \) on \( X \), \( f(\mathcal{F}) \) is said to be the prefilter on \( Y \) generated by \( \{ f(A) : A \in \mathcal{F} \} \).

**DEFINITION 3.2.** ([4]) A fuzzy convergence structure on \( X \) is a function \( C_X \) from \( P(X) \) into \( F(X) \) satisfying the following conditions:

(FC1) For each fuzzy point \( p \) in \( X \), \( p \in C_X(\hat{p}) \).
(FC2) For \( \Phi, \Psi \in P(X) \), if \( \Phi \subseteq \Psi \) then \( C_X(\Phi) \subseteq C_X(\Psi) \).
(FC3) If \( p \in C_X(\Phi) \), then \( p \in C_X(\Phi \cap \hat{p}) \).

Then the pair \( (X, C_X) \) is said to be fuzzy convergence space. If \( p \in C_X(\hat{p}) \), we say that \( \Phi \) \( C_X \)-converges to a fuzzy point \( p \). The prefilter \( V_{C_X}(p) \) obtain by intersecting all prefilters which \( C_X \)-converge to \( p \) is said to be the \( C_X \)-neighborhood prefilter at \( p \). If \( V_{C_X}(p) \) \( C_X \)-
converges to $p$ for each fuzzy point $p$ in $X$, then $C_X$ is called a fuzzy pretopological structure, and $(X, C_X)$ a fuzzy pretopological space. The fuzzy pretopological structure $C_X$ is said to be fuzzy topological structure and $(X, C_X)$ is said to be fuzzy topological space, if for each fuzzy point $p$ in $X$, the prefILTER $N_{C_X}(p)$ has a prefILTER base $\beta_{C_X}(p) \subseteq N_{C_X}(p)$ with the following property:

\[ r \cup \in \beta_{C_X}(p) \quad \text{implies} \quad \cup \in \beta_{C_X}(r) \]

Throughout this paper, let $C(X)$ be the set of all fuzzy convergence structures on $X$. Then we define that $C_1 \leq C_2$ for $C_1, C_2 \in C(X)$ if and only if $C_2(\Phi) \subseteq C_1(\Phi)$ for all $\Phi \in P(X)$. If $C_1 \leq C_2$ for $C_1, C_2 \in C(X)$, we say that $C_2$ is finer than $C_1$, also that $C_1$ is coarser than $C_2$.

**DEFINITION 3.3.** The fuzzy convergence space $(X, C_X)$ is said to be fuzzy Hausdorff if each prefILTER $\mathcal{F}$ $C_X$-converges to at most one fuzzy point $p$ in $X$.

Let $F(X)$ be the set of all fuzzy sets in $X$ and $A$ a fuzzy set in $X$. The set function $\Gamma_{C_X}$ (resp. $I_{C_X}$) from $F(X)$ into $F(X)$ is given by $\Gamma_{C_X}(A) = \{ p : p$ is fuzzy point in $X$ and $p \in C_X(\mathcal{F})$ for some ultra prefILTER $\mathcal{F}$ with $A \in F \}$ (resp. $I_{C_X}(A) = \{ p : A \in N_{C_X}(p)$ and $p$ is a fuzzy point in $X \}$). Then $\Gamma_{C_X}(A)$ (resp. $I_{C_X}(A)$) is called fuzzy closure of fuzzy set $A$ (resp. fuzzy interior of $A$).

For a prefILTER $\mathcal{F}$ on $X$, $\Gamma_{C_X}(\mathcal{F})$ and $I_{C_X}(\mathcal{F})$ are the prefILTERs on $X$ generated by $\{ \Gamma_{C_X}(A) : A \in F \}$ and $\{ I_{C_X}(A) : A \in F \}$, respectively.

**DEFINITION 3.4.** The fuzzy convergence space $(X, C_X)$ is called fuzzy regular (resp. fuzzy semi-regular) if $\Gamma_{C_X}(\mathcal{F})$ (resp. $I_{C_X}(\Gamma_{C_X}(\mathcal{F}))$) $C_X$-converges to $p$, whenever fuzzy prefILTER $\mathcal{F}$ $C_X$-converges to fuzzy point $p$.

Let $(X, C_X)$ be a fuzzy convergence space and $A$ a subset of $X$. $(A, C_A)$ is called a fuzzy convergence subspace of $(X, C_X)$ if each prefILTER $\mathcal{F}$ on $A$ $C_A$-converges to $p$ in $A$ whenever the prefILTER on $X$ generated by $\mathcal{F}$ $C_X$-converges to $p$.

From definition of set functions $\Gamma_{C_X}$ and $I_{C_X}$, we can obtain the followings: $\Gamma_{C_X} \supseteq A$ and $I_{C_X}(A) \subseteq A$ for each fuzzy set $A$ in $X$. 
4. Several continuous functions on fuzzy convergence spaces

In this section, we define super continuity, $\delta$-continuity weakly $\delta$-continuity on fuzzy convergence spaces and investigate the relationships among them.

And we introduce initial fuzzy convergence structure and product fuzzy convergence space.

Through this section, let $(X, C_X)$ and $(Y, C_Y)$ be the fuzzy convergence spaces and $p$ a fuzzy point in $X$.

**Definition 4.1.** A function $f$ from $(X, C_X)$ to $(Y, C_Y)$ is continuous at $p$ if $f(F) C_Y$-converges to $f(p)$, whenever a prefILTER $F$ on $X C_X$-converges to $p$.

**Definition 4.2.** A function $f$ from $X$ to $Y$ is said to be fuzzy super continuous at $p$ in $X$ if $\mathcal{V}_{C_Y}(f(p)) \subseteq f(I_{C_X}(\Gamma_{C_X}(F)))$ whenever a prefILTER $F$ on $X C_X$-converges to $p$.

**Definition 4.3.** A function $f$ from $X$ to $Y$ is said to be fuzzy $\delta$-continuous at $p$ in $X$ if $I_{C_Y}(\Gamma_{C_Y}(\mathcal{V}_{C_Y}(f(p)))) \subseteq f(I_{C_X}(\Gamma_{C_X}(F)))$ whenever a prefILTER $F$ on $X C_X$-converges to $p$.

**Definition 4.4.** A function $f$ from $X$ to $Y$ is said to be fuzzy weakly $\delta$-continuous at $p$ in $X$ if $\Gamma_{C_Y}(\mathcal{V}_{C_Y}(f(p))) \subseteq f(I_{C_X}(\Gamma_{C_X}(F)))$.

If a function $f$ is fuzzy continuous (resp. super continuous, $\delta$-continuous, and weakly $\delta$-continuous) at each fuzzy point in $X$, then $f$ is said to be fuzzy continuous (resp. super continuous, $\delta$-continuous, and weakly $\delta$-continuous) on $X$.

**Theorem 4.5.** If a function $f$ from $(X, C_X)$ to $(Y, C_Y)$ is fuzzy super continuous at fuzzy point $p$ in $X$, then $f$ is fuzzy weakly $\delta$-continuous at $p$ in $X$.

**Proof.** Suppose that a prefILTER $F C_X$-converges to a fuzzy point $p$ in $X$. Then $\mathcal{V}_{C_Y}(f(p)) \subseteq f(I_{C_X}(\Gamma_{C_X}(F)))$ by Definition 4.2. Since $\Gamma_{C_Y}(\mathcal{V}_{C_Y}(f(p))) \subseteq \mathcal{V}_{C_Y}(f(p))$, $\Gamma_{C_Y}(\mathcal{V}_{C_Y}(f(p))) \subseteq f(I_{C_X}(\Gamma_{C_X}(F)))$. Accordingly, $f$ is fuzzy weakly $\delta$-continuous at $p$ in $X$ by Definition 4.4.
THEOREM 4.6. Let \((Y, C_Y)\) be a fuzzy regular pretopological space. If a function \(f\) from \((X, C_X)\) to \((Y, C_Y)\) is fuzzy weakly \(\delta\)-continuous at fuzzy point \(p\) in \(X\), then \(f\) is fuzzy super continuous at \(p\) in \(X\).

Proof. Suppose that a prefilter \(FC_X\)-converges to a fuzzy point \(p\) in \(X\). Then, since \((Y, C_Y)\) is fuzzy regular pretopological space, \(\Gamma_{C_Y}(V_{C_Y}(f(p)))\) \(C_Y\)-converges to \(f(p)\) in \(Y\). Accordingly, by definition of \(V_{C_Y}(f(p))\), \(V_{C_Y}(f(p)) \subseteq \Gamma_{C_Y}(V_{C_Y}(f(p)))\). Since \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\), \(V_{C_Y}(f(p)) \subseteq f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\). Thus \(f\) is fuzzy super continuous at \(p\) in \(X\).

THEOREM 4.7. If a function \(f\) from \((X, C_X)\) to \((Y, C_Y)\) is fuzzy \(\delta\)-continuous at fuzzy point \(p\) in \(X\), then \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\).

Proof. Suppose that a prefilter \(FC_X\)-converges to fuzzy point \(p\) in \(X\). Then \(I_{C_Y}(\Gamma_{C_Y}(V_{C_Y}(f(p)))) \subseteq f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\) by Definition 4.3. Since \(\Gamma_{C_Y}(V_{C_Y}(f(p))) \subseteq I_{C_Y}(\Gamma_{C_Y}(V_{C_Y}(f(p))))\), \(\Gamma_{C_Y}(V_{C_Y}(f(p))) \subseteq f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\). Thus \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\) by Definition 4.4.

THEOREM 4.8. Let \((X, C_X)\) be a fuzzy semi-regular convergence space. If a function \(f\) from \((X, C_X)\) to \((Y, C_Y)\) is fuzzy continuous at a fuzzy point \(p\) in \(X\), then \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\).

Proof. Suppose that a prefilter \(\mathcal{F}\) on \(X\) \(C_X\)-converges to a fuzzy point \(p\) in \(X\). Then, \(I_{C_X}(\Gamma_{C_X}(\mathcal{F}))\) \(C_X\)-converges to \(p\), as \((X, C_X)\) is fuzzy semi-regular convergence space. Since \(f\) is fuzzy continuous at \(p\) in \(X\), \(f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\) \(C_Y\)-converges to \(f(p)\) in \(Y\). Accordingly, \(V_{C_Y}(f(p)) \subseteq f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\). But \(\Gamma_{C_Y}(V_{C_Y}(f(p))) \subseteq V_{C_Y}(f(p))\). Thus \(\Gamma_{C_Y}(V_{C_Y}(f(p))) \subseteq f(I_{C_X}(\Gamma_{C_X}(\mathcal{F})))\), and so \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\).

A fuzzy regular convergence space is fuzzy semi-regular, and so the following corollary is trivial.

COROLLARY 4.9. Let \((X, C_X)\) be a fuzzy regularly convergence space. If a function from \((X, C_X)\) to \((Y, C_Y)\) is fuzzy continuous at a fuzzy point \(p\) is \(X\), then \(f\) is fuzzy weakly \(\delta\)-continuous at \(p\) in \(X\).

Fuzzy \(\delta\)-continuity and fuzzy super continuity are independent. But if \((Y, C_Y)\) is the fuzzy regular pretopological space and the function
Theorem 4.10. If a function $f$ from $(X, C_X)$ to $(Y, C_Y)$ is fuzzy continuous and $A$ is fuzzy set in $X$ then $f(\Gamma_{C_X}(A)) \subseteq \Gamma_{C_Y}(f(A))$.

Proof. If $q \in f(\Gamma_{C_X}(A))$, then there exists a fuzzy point $p$ in $\Gamma_{C_X}(A)$ such that $f(p) = q$. By definition $\Gamma_{C_X}(A)$, there exists an ultra prefilter $F$ such that $p \in C_X(F)$ and $A \in F$. Since $f$ is fuzzy continuous, $f(F)$ converges to $f(p) = q$, and $f(A) \in f(F)$. By definition of $\Gamma_{C_Y}(f(A)), q = f(p) \in \Gamma_{C_Y}(f(A))$. Thus $f(\Gamma_{C_X}(A)) \subseteq \Gamma_{C_Y}(f(A))$.

Theorem 4.11. Let a $f$ from $(X, C_X)$ to $(Y, C_Y)$ be a map, and $p$ any fuzzy point in $X$ and $A$ any fuzzy set in $X$. Then the followings are equivalent

1. $f(V_{C_X}(p)) \subseteq V_{C_Y}(f(p))$.

2. $f(I_{C_X}(A)) \subseteq I_{C_Y}(f(A))$.

Proof. (1) $\implies$ (2) If a fuzzy point $q \in f(I_{C_X}(A))$, then there is a fuzzy point $p \in I_{C_X}(A)$ such that $f(p) = q$. Then $p \in A$ and $A \in V_{C_X}(p)$ by definition of $I_{C_X}(A)$. Accordingly, $f(p) \in f(A)$ and $f(A) \in f(V_{C_X}(p)) \subseteq V_{C_Y}(f(p))$ by (1). Thus, $q = f(p) \in I_{C_Y}(f(A))$ by definition of $I_{C_Y}(f(A))$, and so $f(I_{C_X}(A)) \subseteq I_{C_Y}(f(A))$.

(2) $\implies$ (1) If $B \in f(V_{C_X}(p))$, then there is a fuzzy set $A$ in $V_{C_X}(p)$ such that $f(A) \subseteq B$. Accordingly, $p \in I_{C_X}(A)$ by definition of $I_{C_X}(A)$, and so $f(p) \in f(I_{C_X}(A))$. Since $f(I_{C_X}(A)) \subseteq I_{C_Y}(f(A)) \subseteq I_{C_Y}(B), f(p) \in I_{C_Y}(B)$, and so $B \in V_{C_Y}(f(p))$ by $I_{C_Y}(B)$. Thus $f(V_{C_X}(p)) \subseteq V_{C_Y}(f(p))$.

Theorem 4.12. Let $X$ be a nonempty set, $(X_\lambda, C_{X_\lambda})$ a fuzzy convergence spaces, and $f_\lambda : X \to (X_\lambda, C_{X_\lambda})$ a surjection for each $\lambda \in \Lambda$. If $C_X$ is a map from the set $P(X)$ of all prefilters on $X$ to the set $F(X)$ of all fuzzy sets in $X$ satisfying the following condition (*):

(*) for any fuzzy point $p$ in $X$ and $\Phi \in P(X)$,

$p \in C_X(\Phi)$ if and only if $f_\lambda(\Phi)C_{X_\lambda}$-converges to $f_\lambda(p)$ for each $\lambda \in \Lambda$, then $C_X$ is a fuzzy convergence structure on $X$.

Proof. Let $p$ be any fuzzy point in $X$, then $\hat{f}_\lambda(p) = f_\lambda(\hat{p})C_{X_\lambda}$-converges to $f_\lambda(p)$ for each $\lambda \in \Lambda$. Thus $p \in C_X(\hat{p})$ by hypothesis. If $\Phi$ and $\Psi$ are two prefilters on $X$ with $\Phi \subseteq \Psi$ and $p \in C_X(\Phi)$, then $f_\lambda(\Phi)C_{X_\lambda}$-converges to $f_\lambda(p)$ for each $\lambda \in \Lambda$. Since $f_\lambda(\Phi) \subseteq f_\lambda(\Psi)$
for each $\lambda \in \Lambda$, by definition of $C_{X\lambda}$, $f_\lambda(\Psi)C_{X\lambda}$-converges to $f_\lambda(p)$ for each $\lambda \in \Lambda$. Therefore, $p \in C_X(\Psi)$. Finally, if $\Phi$ is a prefilter on $X$ with $p \in C_X(\Phi)$, then $f_\lambda(\Phi \cap p) = f_\lambda(\Phi) \cap f_\lambda(p)$ converges to $f_\lambda(p)$ for each $\lambda \in \Lambda$. Thus $p \in C_X(\Phi \cap p)$. The fuzzy convergence structure $C_X$, is given in Theorem 4.12, is called the initial convergence structure on $X$ induced by the family $\{f_\lambda : \lambda \in A\}$. 

**Theorem 4.13.** The initial fuzzy convergence structure $C_X$ is the coarsest convergence structure on $X$ which allows $f_\lambda$ to be fuzzy continuous for each $\lambda \in \Lambda$.

**Proof.** By Definition 4.1, it is clear that $f_\lambda$ is continuous for each $\lambda \in \Lambda$. Let $C'_X$ be a fuzzy convergence structure on $X$ relative to which $f_\lambda$ is continuous for each $\lambda \in \Lambda$. Suppose that a prefilter $\Phi$ on $X$ $C'_X$-converges to the fuzzy point $p$ in $X$. Since $f_\lambda : (X, C'_X) \rightarrow (X, C_{X\lambda})$ is continuous for each $\lambda \in \Lambda$, then $f_\lambda(\Phi)C_{X\lambda}$-converges to $f_\lambda(p)$. Accordingly, by definition of the initial fuzzy convergence structure $C_X$, $\Phi C_X$-converges to $p$, that is, $C_X \leq C'_X$. Thus, $C_X$ is the coarsest fuzzy convergence structure on $X$ which allows $f_\lambda$ to be continuous for each $\lambda \in \Lambda$. Let $\{(X_\lambda, C_{X\lambda}) : \lambda \in \Lambda\}$ be a family of fuzzy convergence spaces and $P_\lambda : \Pi X_\lambda \rightarrow (X_\lambda, C_{X\lambda})$ be canonical projection for each $\lambda \in \Lambda$. Then, $\Pi X_\lambda$ endowed with initial fuzzy convergence structure $C_X$ induced by $\{P_\lambda : \lambda \in \Lambda\}$ is called the product fuzzy convergence space induced by $\{(X_\lambda, C_{X\lambda}) : \lambda \in \Lambda\}$. In this case $C_X$ is denoted by $\Pi_{\lambda \in \Lambda} C_{X\lambda}$, that is, $C_X = \Pi_{\lambda \in \Lambda} C_{X\lambda}$.

**Theorem 4.14.** If $\{(\Pi X_\lambda, C_X) : \lambda \in \Lambda\}$ is the product fuzzy convergence space of family $\{(X_\lambda, C_{X\lambda}) : \lambda \in \Lambda\}$ of fuzzy convergence spaces, then

1. $\Pi_{\lambda \in \Lambda} C_{X\lambda}(p_\lambda) \subseteq C_X(p)$,

2. $P_{\lambda}(C_X(p)) = \Pi_{\lambda \in \Lambda} C_{X\lambda}(p_\lambda)$ for each $\lambda \in \Lambda$,

where $p = (p_\lambda)_{\lambda \in \Lambda}$ is the fuzzy point in $\Pi_{\lambda \in \Lambda} X_\lambda$ for fuzzy point $p_\lambda$ in $X_\lambda$ and $P_{\lambda} : \Pi_{\lambda \in \Lambda} X_\lambda \rightarrow X_\lambda$ canonical projection for each $\lambda \in \Lambda$.

**Proof.** (1) If $F \in \Pi_{\lambda \in \Lambda} C_{X\lambda}(p_\lambda)$, then there exists a $F_\lambda = \Pi_{\lambda \in \Lambda} F_\lambda$ such that $F_\lambda \subseteq F$, where $F_\lambda \in C_{X\lambda}(p_\lambda)$ for each $\lambda \in \Lambda$ and $F_\lambda = X_\lambda$ for all but a finite number of indices. Suppose that $F_\lambda \not\subseteq C_X(p)$, then there
exists a prefILTER $\Phi_{C_X}$-converges to $p$ such that $F_\infty \notin \Phi$. Accordingly, $P_\lambda(\Phi)C_{X,\lambda}$-converges to $P_\lambda(p) = p_\lambda$ for each $\lambda \in \Lambda$, and so $\mathcal{V}_{C_{X,\lambda}}(p_\lambda) \subseteq P_\lambda(\Phi)$ for each $\lambda \in \Lambda$. Therefore $F_\infty \in \prod_{\lambda \in \Lambda} \mathcal{V}_{C_{X,\lambda}}(p_\lambda) \subseteq \prod_{\lambda \in \Lambda} P_\lambda(\Phi) \subseteq \Phi$. This contradicts that $F_\infty \notin \Phi$. Thus, $F_\infty \in \mathcal{V}_{C_X}(p)$, and so $F \in \mathcal{V}_{C_X}(p)$. Consequently $\prod_{\lambda \in \Lambda} \mathcal{V}_{C_{X,\lambda}}(p_\lambda) \subseteq \mathcal{V}_{C_X}(p)$.

(2) Since $P_\lambda$ is continuous, $\mathcal{V}_{C_{X,\lambda}}(p_\lambda) \subseteq P_\lambda(\mathcal{V}_{C_X}(p))$ for each $\lambda \in \Lambda$. Now, we will show that $\mathcal{V}_{C_{X,\lambda}}(p_\lambda) \supseteq P_\lambda(\mathcal{V}_{C_X}(p))$. If $F_\mu$ is any element of $P_\mu(\mathcal{V}_{C_X}(p))$, then there is a $F \in \mathcal{V}_{C_X}(p)$ such that $P_\mu(F) \subseteq F_\mu$. Let $F$ be an arbitrary prefILTER which $C_{X,\mu}$-converges to $p_\mu$.

Take the prefILTER $\prod_{\lambda \in \Lambda} \Phi_\lambda$ on $\prod_{\lambda \in \Lambda} X_\lambda$, where

$$\Phi_\lambda = \begin{cases} F, & \text{if } \lambda = \mu \\ p_\lambda, & \text{otherwise.} \end{cases}$$

Then $\prod_{\lambda \in \Lambda} \Phi_\lambda C_X$-converges to $p = (p_\lambda)_{\lambda \in \Lambda}$. Since $F \in \mathcal{V}_{C_X}(p) \subseteq \prod_{\lambda \in \Lambda} \Phi_\lambda$, $P_\mu(F) \in P_\mu(\prod_{\lambda \in \Lambda} \Phi_\lambda) = F$. Therefore $F_\mu \in F$, that is, $F_\mu \in \mathcal{V}_{C_{X,\mu}}(p_\mu)$. Consequently, $\mathcal{V}_{C_{X,\lambda}}(p_\lambda) = P_\lambda(\mathcal{V}_{C_X}(p))$.

**Theorem 4.15.** Let $f_\lambda$ be a map from a fuzzy convergence space $(X_\lambda, C_{X,\lambda})$ to a fuzzy convergence space $(Y_\lambda, C_{Y,\lambda})$ and $P_\lambda$ the canonical projection from $\prod_{\lambda \in \Lambda} X_\lambda$ to $(X_\lambda, C_{X,\lambda})$ for each $\lambda \in \Lambda$. If $f_\lambda$ is fuzzy super continuous for each $\lambda \in \Lambda$, then $f_\lambda \circ P_\lambda$ is fuzzy super continuous.

**Proof.** Suppose that a prefILTER $\mathcal{F}$ on $\prod_{\lambda \in \Lambda} X_\lambda$-$C_{X,\lambda}$-converges to $p = (p_\lambda)_{\lambda \in \Lambda}$. Then, $P_\lambda(\mathcal{F})C_{X,\lambda}$-converges to $P_\lambda(p) = p_\lambda$ in $X_\lambda$. Since $f_\lambda$ is super continuous,

$$\mathcal{V}_{C_{Y,\lambda}}(f_\lambda(P_\lambda(p)) \subseteq f_\lambda(I_{C_{X,\lambda}}(\Gamma_{C_{X,\lambda}}(P_\lambda(\mathcal{F})))).$$

Since $P_\lambda$ is continuous, $I_{C_{X,\lambda}}(\Gamma_{C_{X,\lambda}}(P_\lambda(\mathcal{F}))) \subseteq I_{C_{X,\lambda}}(P_\lambda(\Gamma_{\Pi C_{X,\lambda}}(\mathcal{F})))$ by Theorem 4.10. By Theorem 4.11 and 4.14,

$$I_{C_{X,\lambda}}(P_\lambda(\Gamma_{\Pi C_{X,\lambda}}(\mathcal{F}))) \subseteq P_\lambda(I_{\Pi C_{X,\lambda}}(\Gamma_{\Pi C_{X,\lambda}}(\mathcal{F}))).$$

Thus, $\mathcal{V}_{C_{Y,\lambda}}(f_\lambda(P_\lambda(p))) \subseteq f_\lambda(P_\lambda(I_{\Pi C_{X,\lambda}}(\Gamma_{\Pi C_{X,\lambda}}(\mathcal{F}))))$, and so $f_\lambda \circ P_\lambda$ is super continuous.
THEOREM 4.16. Let f be a map from a fuzzy convergence space \((X, C_X)\) into a fuzzy pretopological convergence space \((Y, C_Y)\) and g a map from \((Y, C_Y)\) into a fuzzy convergence space \((Z, C_Z)\). If f is fuzzy super continuous and g is fuzzy continuous, then the composition \(g \circ f\) is fuzzy super continuous.

Proof. Let a prefilt\(\mathcal{F}\)er \(\mathcal{F}\) on \(X\) \(C_X\)-converges to a fuzzy point \(p\) in \(X\). Since \((Y, C_Y)\) is fuzzy pretopological convergence space \(\mathcal{V}_{C_Y}(f(p))\) \(C_Y\)-converges to \(f(p)\). By definition of fuzzy super continuous, \(\mathcal{V}_{C_Y}(f(p)) \subseteq f(\Gamma_{C_X}(\mathcal{F})))\), and so \(g(\mathcal{V}_{C_Y}(f(p))) \subseteq g(f(\Gamma_{C_X}(\mathcal{F})))\). Since \(g\) is fuzzy continuous, \(g(\mathcal{V}_{C_Y}(f(p)))\) \(C_Z\)-converges to \(g(f(p))\). Thus, \(g(f(\Gamma_{C_X}(\mathcal{F})))\) \(C_Z\)-converges to \(g(f(p))\). Consequently,

\[\mathcal{V}_{C_Z}(g(f(p))) \subseteq g(f(\Gamma_{C_X}(\mathcal{F})))\]

that is, \(g \circ f\) is fuzzy super continuous.

From above Theorem, we obtain the following corollaries.

COROLLARY 4.17. Let \((X_\lambda, C_{X_\lambda})\) be the fuzzy pretopological convergence space for each \(\lambda \in \Lambda\) and \((\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} C_{X_\lambda})\) the fuzzy product convergence space of the family \(\{(X_\lambda, C_{X_\lambda}) : \lambda \in \Lambda\}\) of fuzzy convergence spaces. If \(f : (X, C_X) \to (\prod_{\lambda \in \Lambda} X_\lambda, \prod_{\lambda \in \Lambda} C_{X_\lambda})\) is super continuous, then \(P_\lambda \circ f\) is also super continuous for each \(\lambda \in \Lambda\), where \(P_\lambda : \prod_{\lambda \in \Lambda} X_\lambda \to X_\lambda\) is the canonical projection for each \(\lambda \in \Lambda\).

COROLLARY 4.18. Let \(g : (X, C_X) \to (Y, C_Y)\) be a function and \(f : (X, C_X) \to (X, C_X) \times (Y, C_Y)\) given by \(f(x) = (X, g(x))\) be its graph. If \(f\) is super continuous, then \(g\) is super continuous.

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