WEAK CONVERGENCE TO FIXED POINTS OF ALMOST-ORBITS OF NONLIPSCHITZIAN SEMIGROUPS

TAE-HWA KIM, MAN-DONG HUR AND NAK-EUN CHO

1. Introduction

Let $G$ be a semitopological semigroup, i.e., $G$ is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from $G$ to $G$ are continuous. $G$ is called right reversible if any two closed left ideals of $G$ have nonvoid intersection. In this case, $(G, \geq)$ is a directed system when the binary relation "$\geq$" on $G$ is defined by

$$t \geq s \quad \text{if and only if} \quad \{s\} \cup \bar{G}s \supseteq \{t\} \cup \bar{G}t, \quad s, t \in G.$$ 

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [4, p.335]). Left reversibility of $G$ is defined similarly. $G$ is called reversible if it is both left and right reversible.

Let $G$ be a semitopological semigroup with a binary relation "$\geq$" which directs $G$. Let $C$ be a nonempty closed convex subset of a real Banach space $E$ and let a family $\mathcal{S} = \{S(t) : t \in G\}$ be a (continuous) representation of $G$ as continuous mappings on $C$ into $C$, i.e., $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$, and for every $x \in C$, the mapping $t \mapsto S(t)x$ from $G$ into $C$ is continuous. In this paper, we also consider a non-Lipschitzian semigroup of continuous mappings : a representation $\mathcal{S} = \{S(t) : t \in G\}$ of $G$ on $C$ is said to be a semigroup of asymptotically nonexpansive type (simply, a.n.t.) on $C$ if, for each $x \in C$,

$$\inf_{s \in G} \sup_{t \geq s} \sup_{y \in C} (\|S(t)x - S(t)y\| - \|x - y\|) \leq 0.$$ 

Received March 30, 1992.
Immediately, we can see that the semigroups of a.n.t. include all semigroups of nonexpansive mappings with directed systems. \( \mathcal{G} = \{S(t) : t \in G\} \) is called reversible [resp., right(left) reversible] if \( G \) is reversible [resp., right(left) reversible]. For a mapping \( S : C \to C \), we define \( S(n) = S^n \) for each \( n \in G = N \), where \( N \) denotes the set of natural numbers. Then, when the semigroup \( \mathcal{G} = \{S(n) : n \in G\} \) is of a.n.t., the mapping \( S : C \to C \) is simply said to be of a.n.t. In particular, if \( \mathcal{G} = \{S(t) : t \in G\} \) is a Lipschitzian representation of \( G \) with an additional condition, i.e., \( \limsup_t k_t \leq 1 \) (see [7]), and if \( C \) is bounded, then it is obviously of a.n.t. And we say that a function \( u : G \to C \) is an almost-orbit of \( \mathcal{G} = \{S(t) : t \in G\} \) (see [1], [7]) if \( G \) is right reversible and

\[
\lim_{t} \left( \sup_{s \in G} \|u(st) - S(s)u(t)\| \right) = 0.
\]

In [7], Takahashi-Zhang established the weak convergence of an almost-orbit of a noncommutative Lipschitzian semigroup in a Banach space. And in [6], Lau-Takahashi proved the nonlinear ergodic theorems for a noncommutative nonexpansive semigroup in the space. In this paper, we shall establish the weak convergence to a fixed point of an almost-orbit \( \{u(t) : t \in G\} \) of the right reversible semigroup \( \mathcal{G} = \{S(t) : t \in G\} \) of a.n.t. in a uniformly convex Banach space with a Fréchet differentiable norm, which extends the result according to Takahashi-Zhang [7].

2. Preliminaries and some lemmas

Let \( C \) be a nonempty closed convex subset of a real Banach space \( E \) and let \( G \) be a semitopological semigroup with a binary relation "\( \preceq \)" which directs \( G \). A family \( \mathcal{G} = \{S(t) : t \in G\} \) of continuous mappings from \( C \) into itself is said to be a (continuous) representation of \( G \) on \( C \) if \( \mathcal{G} \) satisfies the following:

(a) \( S(ts)x = S(t)S(s)x \) for all \( t, s \in G \) and \( x \in C \);

(b) for every \( x \in C \), the mapping \( t \mapsto S(t)x \) from \( G \) into \( C \) is continuous. A representation \( \mathcal{G} = \{S(t) : t \in G\} \) of \( G \) on \( C \) is said to be a semigroup of asymptotically nonexpansive type (simply, a.n.t.) on \( C \) if, for each \( x \in C \),

\[
\inf_{s \in G} \sup_{t \succeq s} \sup_{y \in C} \left( \|S(t)x - S(t)y\| - \|x - y\| \right) \leq 0.
\]
Immediately, we can see that the semigroups of a.n.t. include all semigroups of nonexpansive mappings with directed systems. In particular, if \( \mathcal{S} = \{ S(t) : t \in G \} \) is a Lipschitzian representation of \( G \) with an additional condition, i.e., \( \limsup k_t \leq 1 \) (see [7]), and if \( C \) is bounded, then it is obviously of a.n.t.

Let \( G \) be a semitopological semigroup, i.e., \( G \) is a semigroup with a Hausdorff topology such that for each \( a \in G \) the mappings \( s \mapsto a \cdot s \) and \( s \mapsto s \cdot a \) from \( G \) to \( G \) are continuous. \( G \) is called right reversible if any two closed left ideals of \( G \) have nonvoid intersection. In this case, \((G, \unrhd)\) is a directed system when the binary relation \( \unrhd \) on \( G \) is defined by

\[
t \unrhd s \quad \text{if and only if} \quad \{s\} \cup \overline{Gs} \supseteq \{t\} \cup \overline{Gt}, \quad s, t \in G.
\]

Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups (see [4, p.335]). Left reversibility of \( G \) is defined similarly. \( G \) is called reversible if it is both left and right reversible.

In particular, if \( G \) is right reversible, and if \( \mathcal{S} = \{ S(t) : t \in G \} \) is a semigroup of a.n.t. on \( C \), then a function \( u : G \rightarrow C \) is called an almost-orbit of \( \mathcal{S} = \{ S(t) : t \in G \} \) if

\[
\lim_{t \to 0} \left( \sup_{s} \| u(st) - S(s)u(t) \| \right) = 0.
\]

With each \( x \in E \), we associate the set

\[
J(x) = \{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \}.
\]

Using the Hahn-Banach theorem it is immediately clear that \( J(x) \neq \emptyset \) for any \( x \in E \). The multivalued operator \( J : E \rightarrow 2^{E^*} \) is called the duality mapping of \( E \). The norm of \( E \) is said to be Gâteaux differentiable (and \( E \) is said to be smooth) if for each \( x, y \in S \),

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists, where \( S \) denotes the unit sphere of \( E \). It is said to be Fréchet differentiable if, for each \( x \in S \), this limit (2.4) is attained uniformly for
Then it is well-known that if $E$ is smooth, the duality mapping $J$ is single-valued. We also know that if $E$ has a Fréchet differentiable norm, then $J$ is norm to norm continuous.

In order to measure the degree of strict convexity (rotundity) of $E$, we define its modulus of convexity $\delta : [0, 2] \to [0, 1]$ by

$$\delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} \| x + y \| : \| x \| \leq 1, \| y \| \leq 1, \text{ and } \| x - y \| \geq \varepsilon \right\}.$$  

The characteristic of convexity $\varepsilon_\ast$ of $E$ is also defined by

$$\varepsilon_\ast = \varepsilon_\ast(E) = \sup \{ \varepsilon : \delta(\varepsilon) = 0 \}.$$  

It is well-known (see [3]) that the modulus of convexity $\delta$ satisfies the following properties:

$$(a) \quad \delta \text{ is increasing on } [0, 2], \text{ and moreover strictly increasing on } [\varepsilon_\ast, 2];$$

$$(b) \quad \delta \text{ is continuous on } [0, 2] \text{ (but not necessarily at } \varepsilon = 2);$$

$$(c) \quad \delta(2) = 1 \text{ if and only if } E \text{ is strictly convex;}$$

$$(d) \quad \delta(0) = 0 \text{ and } \lim_{\varepsilon \to 2-} \delta(\varepsilon) = 1 - \frac{1}{2} \varepsilon_\ast;$$

$$(e) \quad \| a - x \| \leq r, \| a - y \| \leq r \text{ and } \| x - y \| \geq \varepsilon \Rightarrow \| a - \frac{1}{2}(x + y) \| \leq r(1 - \delta(\varepsilon/r)).$$

A Banach space $E$ is said to be uniformly convex if $\delta(\varepsilon) > 0$ for all positive $\varepsilon$; equivalently $\varepsilon_\ast = 0$. Obviously, any uniformly convex space is both strictly convex and reflexive. By properties above, we can see that if $E$ is uniformly convex, then $\delta$ is strictly increasing and continuous on $[0, 2]$ (see [2]).

It is easy that if $G$ is right reversible and $u = \{ u(t) : t \in G \}$ is an almost-orbit of the semigroup $\mathcal{G} = \{ S(t) : t \in G \}$ of a.n.t., then $F(\mathcal{G}) \subseteq E(u)$, where $E(u) = \{ y \in C : \lim_{t \to t_0} \| u(t) - y \| \text{ exists} \}$ and $F(\mathcal{G})$ denotes the set of all common fixed points of $\mathcal{G}$.  

**Lemma 2.1.** Let $C$ be a nonempty closed convex of a uniformly convex Banach space $E$. Let $G$ be right reversible and let $\mathcal{G} = \{ S(t) : t \in G \}$ be a semigroup of a.n.t. on $C$. Let $u = \{ u(t) : t \in G \}$ be an almost-orbit of $\mathcal{G}$. Suppose $F(\mathcal{G}) \neq \emptyset$ and let $y \in F(\mathcal{G})$ and $0 < \alpha \leq \beta < 1$. Then, for any $\varepsilon > 0$, there is $t_\ast \in G$ such that

$$\| S(t_\ast)(\lambda u(s) + (1 - \lambda)y) - (\lambda S(t_\ast)u(s) + (1 - \lambda)y) \| < \varepsilon$$
for all $t, s \geq t_*$ and $\lambda \in [\alpha, \beta]$.

Proof. Let $\epsilon > 0$, $c = \min \{2\lambda(1 - \lambda) : \alpha \leq \lambda \leq \beta\}$ and let $r = \lim ||u(t) - y||$. If $r = 0$, since $\mathcal{S} = \{S(t) : t \in G\}$ is of a.n.t. on $C$, there exists $t_* \in G$ such that

$$||y - S(t)z|| < ||y - z|| + \frac{\epsilon}{4}$$

and

$$||u(t) - y|| < \frac{\epsilon}{4} \quad \text{for } t \geq t_* \text{ and } z \in C.$$  

Hence, for $s, t \geq t_*$ and $0 \leq \lambda \leq 1$,

$$||S(t)(\lambda u(s) + (1 - \lambda)y) - (\lambda S(t)u(s) + (1 - \lambda)y)||$$

$$\leq ||S(t)(\lambda u(s) + (1 - \lambda)y) - y|| + \lambda||S(t)u(s) - y||$$

$$\leq 2(||u(s) - y|| + \frac{\epsilon}{4}) < \epsilon.$$  

Now, let $r > 0$. Then we can choose $d > 0$ so small that

$$(r + d)[1 - c\delta(\frac{\epsilon}{r + d})] = r_* < r,$$

where $\delta$ is the modulus of convexity of $E$. On taking $a > 0$ with $a < \min \{\frac{d}{2}, \frac{r - r_*}{2}\}$, there exists $t_* \in G$ such that

(2.8) $r - a < ||u(t) - y|| < r + a,$

(2.9) $||y - S(t)z|| < ||y - z|| + \frac{c}{4}d,$

and

(2.10) $||u(st) - S(s)u(t)|| \leq a,$

for all $t \geq t_*, s \in G$ and $z \in C$. Suppose that

$$||S(t)(\lambda u(s) + (1 - \lambda)y) - (\lambda S(t)u(s) + (1 - \lambda)y)|| \geq \epsilon$$
for some $s,t \geq t_*$ and $\lambda \in [\alpha, \beta]$. Put $z = \lambda u(s) + (1 - \lambda)y$, $u = (1 - \lambda)(S(t)z - y)$ and $v = \lambda(S(t)u(s) - S(t)z)$. Then, by (2.8) and (2.9), we have

$$
\|u\| \leq (1 - \lambda)(\|y - z\| + \frac{c}{4}d)
= (1 - \lambda)(\lambda\|u(s) - y\| + \frac{c}{4}d)
< (1 - \lambda)(\lambda(r + \frac{d}{2}) + \frac{c}{4}d)
< \lambda(1 - \lambda)(r + d)
$$

and

$$
\|v\| < \lambda(1 - \lambda)(r + d).
$$

We also have that

$$
\|u - v\| = \|S(t)z - (\lambda S(t)u(s) + (1 - \lambda)y)\| \geq \varepsilon
$$

and $\lambda u + (1 - \lambda)v = \lambda(1 - \lambda)(S(t)u(s) - y)$. So, by uniform convexity of $E$, we have

$$
\lambda(1 - \lambda)\|S(t)u(s) - y\| = \|\lambda u + (1 - \lambda)v\|
\leq \lambda(1 - \lambda)(r + d)[1 - 2\lambda(1 - \lambda)\delta(\frac{\varepsilon}{r + d})]
\leq \lambda(1 - \lambda)r_*,
$$

and hence $\|S(t)u(s) - y\| \leq r_*$. Then, it follows from (2.10) that

$$
\|u(t_*) - y\| \leq \|u(t_*) - S(t)u(s)\| + \|S(t)u(s) - y\|
< \alpha + r_* < r - \alpha,
$$

which contradicts to (2.8) and the proof is complete.

For $z, y \in E$, we denote by $[x, y]$ the set $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$. The following lemma was proved by Lau-Takahashi [6, Lemma 3].

**Lemma 2.2.** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm and let $\{x_\alpha\}$ be a bounded net in $C$. Let $z \in \bigcap_{\beta}cl\{x_\alpha : \alpha \geq \beta\}$, $y \in C$ and $\{y_\alpha\}$ a net of elements in $C$ with $y_\alpha \in [y, x_\alpha]$ and

$$
\|y_\alpha - z\| = \min\{\|u - z\| : u \in [y, x_\alpha]\}.
$$

If $y_\alpha \to y$, then $y = z$. 
3. Weak convergence theorem

In this section, we study the weak convergence of an almost-orbit \( \{u(t) : t \in G\} \) of \( \mathcal{G} = \{S(t) : t \in G\} \) in a uniformly convex Banach space \( E \) with a Fréchet differentiable norm. By using Lemma 2.1 and Lemma 2.2, we obtain the similar result as Theorem 2 of [6] for the semigroup \( \mathcal{S} := \{S(t) : t \in G\} \) of asymptotically nonexpansive type.

**THEOREM 3.1.** Let \( E \) be a uniformly convex Banach space with a Fréchet differentiable norm and let \( C \) be a nonempty closed convex subset of \( E \). Let \( G \) be right reversible and let \( \mathcal{G} = \{S(t) : t \in G\} \) be a semigroup of a.n.t. on \( C \). Suppose that \( u = \{u(t) : t \in G\} \) is an almost-orbit of \( \mathcal{G} \) and \( F(\mathcal{G}) \neq \emptyset \). Then the set \( \bigcap_s \overline{co}\{u(t) : t \geq s\} \cap F(\mathcal{G}) \)

consists of at most one point.

**Proof.** Let \( W(u) = \bigcap_s \overline{co}\{u(t) : t \geq s\} \). Suppose that \( f, g \in W(u) \cap F(\mathcal{G}) \) and \( f \neq g \). Put \( h = (f + g)/2 \) and \( r = \liminf_s \|u(s) - g\| \) by Lemma 2.1. Since \( h \in W(u) \), we have \( \|h - g\| \leq r \). For each \( s \in G \), choose \( p_s \in [u(s), h] \) such that

\[
\|p_s - g\| = \min\{\|y - g\| : y \in [u(s), h]\}.
\]

By the definition of \( p_s \), we have \( \|p_s - g\| \leq \|(p_s + h)/2 - g\| \leq \|h - g\| \) for all \( s \in G \). If \( \liminf_s \|p_s - g\| = \|h - g\| \), then \( \{p_s\} \) converges strongly to \( h \). Hence, by Lemma 2.2, we have \( h = g \). This contradicts \( f \neq g \).

To complete the proof, we suppose that

\[
\liminf_s \|p_s - g\| < \|h - g\|.
\]

Then there exist \( c > 0 \) and \( t_\alpha \in G \) such that \( t_\alpha \geq \alpha \) and

\[
\|p_{t_\alpha} - g\| + c < \|h - g\| \quad \text{for every} \quad \alpha \in G.
\]

Put \( p_{t_\alpha} = a_\alpha u(t_\alpha) + (1 - a_\alpha)h \) for every \( \alpha \). Then there is \( \beta > 0 \) and \( \gamma < 1 \) such that \( \beta \leq a_\alpha \leq \gamma \) for every \( \alpha \). By (2.1), (2.2), and Lemma 2.1, there exists \( \alpha_0 \in G \) such that

\[
\|g - S(s)z\| < \frac{c}{3} + \|g - z\|,
\]

\[
\|u(st) - S(s)u(t)\| < \frac{c}{3},
\]

and
\[ \|S(s)(\lambda u(t) + (1 - \lambda)h) - (\lambda S(s)u(t) + (1 - \lambda)h)\| < \frac{\epsilon}{3}, \]

for all \( s, t \geq \alpha_o, \ z \in C \) and \( \lambda \in [\beta, \gamma]. \) For \( s \geq \alpha_o, \) since \( t_{\alpha_o} \geq \alpha_o, \) the above inequalities imply that

\[ \|p_{st_{\alpha_o}} - g\| \leq \|a_{\alpha_o}u(st_{\alpha_o}) + (1 - a_{\alpha_o})h - g\| \]
\[ \leq a_{\alpha_o}\|u(st_{\alpha_o}) - S(s)u(t_{\alpha_o})\| + \|S(s)p_{t_{\alpha_o}} - (a_{\alpha_o}S(s)u(t_{\alpha_o}) + (1 - a_{\alpha_o})h)\| + \|S(s)p_{t_{\alpha_o}} - g\| < \|h - g\|. \]

Let \( \beta_o = \alpha_o t_{\alpha_o} \) and \( t \geq \beta_o. \) Then, since \( G \) is right reversible, \( t \in \{ \beta_o \} \cup G \beta_o, \) we may assume \( t \in G \beta_o. \) Let \( \{t_o\} \) be a net in \( G \) such that \( t_o \beta_o \to t. \) Then, \( t = st_{\alpha_o}, \ s = \lim_{o} t_{o} \alpha_o \in G \alpha_o \) and hence \( s \geq \alpha_o. \)

Therefore, we obtain \( \|p_t - g\| < \|h - g\| \) for all \( t \geq \beta_o. \) So, we have \( p_t \neq h \) for all \( t \geq \beta_o. \) Now let \( t \geq \beta_o \) and \( u_k = k(h - p_t) + p_t \) for all \( k \geq 1. \) Then \( \|u_k - g\| \geq \|h - g\| \) for all \( k \geq 1 \) and hence \( \langle h - u_k, J(g - h) \rangle \geq 0 \) for all \( k \geq 1, \) where \( J \) is the duality mapping of \( X \) and \( \langle x, f \rangle \) denotes the value of \( f \in X^* \) at \( x \in X. \) Then, since \( p_t \in [u(t), h], \) it easily follows that \( \langle h - u(t), J(g - h) \rangle \leq 0 \) for all \( t \geq \beta_o. \) Immediately, we obtain \( \langle h - y, J(g - h) \rangle \leq 0 \) for all \( y \in \overline{\alpha_o}\{u(t) : t \geq \beta_o\}, \) and hence \( h = g. \) This contradicts \( f \neq g \) and so the proof is complete.

As a direct consequence, we present the following weak convergence of an almost-orbit \( \{u(t) : t \in G\}. \)

**Theorem 3.2.** Let \( E \) be a uniformly convex Banach space with a Fréchet differentiable norm and let \( C \) be a nonempty closed convex subset of \( E. \) Let \( G \) be right reversible and \( \mathfrak{S} = \{S(t) : t \in G\} \) be a semigroup of a.n.t. on \( C. \) Suppose that \( F(\mathfrak{S}) \neq \emptyset \) and let \( \{u(t) : t \in G\} \) be an almost-orbit of \( \mathfrak{S}. \) If \( \omega_w(u) \subseteq F(\mathfrak{S}), \) then the net \( \{u(t) : t \in G\} \) converges weakly to an element of \( F(\mathfrak{S}). \)

**Proof.** Be similar to Theorem 3 of [7].

**References**

Weak convergence to fixed points of almost-orbits


Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608–737, Korea.