STABILIZABILITY AND CONTROL PROPERTIES FOR AN EVOLUTION EQUATION

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1. Introduction

The main results of this paper are to derive an estimate for \( A_t^s \) where \( s \in \mathcal{R} \) and \( A \) is the realization of an elliptic operator with Dirichlet boundary condition and concerned with some control results of the following equation

\[
\begin{align*}
\begin{cases}
  u'(t) + Au(t) &= \Phi f(t), \quad 0 \leq t \leq T \\
  u(0) &= 0
\end{cases}
\end{align*}
\]

where \(-A\) is the infinitesimal generator of an analytic semigroup in a complex Banach space \( X \). The space \( X \) is called \( \delta \)-convex if there exists a real valued function \( \delta \) on \( X \times X \) having the properties

\[
\begin{align*}
  \delta(x, \cdot) &\text{ is convex for each } x \in X \\
  \delta(x,y) &= \delta(y,x) \\
  \delta(x,y) &\leq |x + y| \quad \text{if } |x| \leq 1 \leq |y|
\end{align*}
\]

and \( \delta(0,0) > 0 \).

For example, the sobolev space \( L^p(\Omega) \) and \( l^p \) are \( \delta \)-convex for \( 1 < p < \infty \), while \( L^1(\mathcal{R}, \mathcal{R}) \) and \( l^p \) are not.

In [9], R. Scely established a similar results by estimating for the \( L^p \) norms of the complex power \( AB^z \) where \( AB \) is an elliptic operator \( A \) whose domain is defined by well posed boundary condition \( Bu = 0 \).

From the R. Scely's estimate of \( AB^{x+y} \) for \( x < 0 \), we now can derive the estimate of \( A_t^s \) for \( s \in \mathcal{R} \) where \( A \) is generalized second order elliptic operator and for any \( f \in L^r(0,T; L^p(\Omega)) \) and \( 1 < r < \infty \) the equation (1.1) has a solution \( u \in W^{1,r}(0,T; W^{1,p}_0) \cap L^r(0,T; D(A)) \)

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where $W^{m,p}(\Omega)$ is the set of all functions whose derivative up to degree $m$ in distribution sense belong to $L^p(\Omega)$ and the closure of $C^m_0(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W^{m,p}(\Omega)$.

In section 3, we consider a necessary and sufficient condition for controllability for the general evolution equation in reflexive Banach space $X$. The criteria for controllability can be stated in terms of $A^*$, the adjoint operator of the infinitesimal generator $A$.

We also derive to the relation between stabilizability of solution and the controllability for the equation (1.1).

In section 4, we give the example of retarded system. In this case many author's have discussed these concepts for retarded and neutral systems [3, 5, 7].

In this note, we deal with also the stabilizability of retarded case, and the relation between stabilizable and controllable.

2. The group property of $A^{1s}$

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$. If we set

\begin{equation}
A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial}{\partial x_j}) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c
\end{equation}

where $a_{ij} = a_{ji} \in C^1(\Omega)$ and $a_{ij}(x)$ is positive definite uniformly in $\Omega$, $b_i \in C^1(\Omega)$ and $c \in L^\infty(\Omega)$, then the dual operator $A'$ of $A$ is

\begin{equation}
A' = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (b_i \cdot) + \bar{c}
\end{equation}

Let $A_p$ be $L^p(\Omega)$-realization with boundary value problem, that is,

\begin{equation}
\mathcal{D}(A_p) = \left\{ \begin{array}{ll}
W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) = \{ u \in W^{2,p}(\Omega) : u|_{\partial \Omega} = 0 \} & \text{if } 1 < p < \infty \\
\{ u ; u \in W^{1,p}(\Omega), \quad Au \in L^1(\Omega) \text{ for } 1 \leq q \leq \frac{n}{n-1} \} & \text{if } p = 1
\end{array} \right.
\end{equation}

and $A_p u = Au$ for $u \in \mathcal{D}(A_p)$, then $-A_p$ generates an analytic semi-groups in $L^p(\Omega)$. 
Similarly, we define

\[ D(A_p') = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega), \quad \frac{1}{p} + \frac{1}{p'} = 1 \]

\[ A_p' u = A'u \]

for \( u \in D(A_p') \), then \(-A_p'\) also generates an analytic semigroup in \( L^{p'}(\Omega) \).

We consider the regular Dirichlet boundary problem, now.

An elliptic Dirichlet boundary value problem is regular problem if its system has the smoothness assumptions on the domain and the coefficient introduced above all.

For the sake of simplicity, we assume that \( 0 \in \rho(A_p) \) and \( 0 \in \rho(A'_p) \). Since \(-A_p\) generates an analytic semigroup, there exists \( w \in (0, \frac{\pi}{2}) \) such that

\[(2.4) \quad \Sigma = \{ \lambda : w \leq \arg \lambda \leq 2\pi - r \} \subset \rho(A_p).\]

We can define

\[ (A_p)^z = \frac{1}{2\pi i} \int_{\Gamma} \lambda^z(\lambda - A_p)^{-1} d\lambda, \quad Re z < 0 \]

where the path \( \Gamma \) runs in the resolvent set of \( A_p \) from \( \infty e^{i\theta} \) to \( \infty e^{-i\theta} \), \( w < \theta < \pi \).

In \((a)\), it is established that there exists a constant \( c \) such that

\[ \| A_p^{s+i\gamma} \|_p \leq c c^{\gamma(x)}, \quad x < 0 \]

where \( c \) depends on \( A_p \), \( \rho \) and \( \gamma \) is the constant in \( (2.4) \).

**Lemma 2.1.** \( A_p^{s}, \ s \in \mathcal{R} \) is a bounded if and only if \( A_p^{-\epsilon+i\gamma} \) is a bounded for any \( \epsilon > 0 \).

**Proof.** Let \( A^{s} \) be a bounded for any \( s \in \mathcal{R} \). It is known that there exists a constant \( M > 0 \) such that

\[ \| A^{-\epsilon} \| \leq M \]

for \( 0 \leq \epsilon \leq 1 \).
For any $x \in \mathcal{D}(A)$ and $\alpha > 0$.

$$A^{-\alpha}x - x = A^{-\alpha}A^{-1} Ax - x = (A^{-\alpha -1} - A^{-1})Ax \to 0$$
as $\alpha \to 1$ ($\alpha \downarrow 1$).

This shows that $A^{-\alpha} \to I$ strongly as $\alpha \to 0$, therefore it follows that for any $\varepsilon > 0$.

$$\|A^{-\varepsilon}\| \leq M$$

for the sufficiently large constant $M > 0$.

We have

$$\|A^{-\varepsilon + i s}\| = \|A^{-\varepsilon}A^{i s}\| \leq M \|A^{i s}\|$$

where $\|A^{-\varepsilon}\| \leq M$

Hence, $A^{-\varepsilon + i s}$ is a bounded for each $\varepsilon > 0$.

Conversely, for any $\varepsilon > 0$, let $\|A^{-\varepsilon + i s}\|$ be bounded, that is $\|A^{-\varepsilon + i s}\| \leq c$. For any $x \in \mathcal{D}(A^{i s})$

$$A^{-\varepsilon + i s}x = A^{-\varepsilon}A^{i s}x \to A^{i s}x$$
as $\varepsilon \to 0$. For $x \in L^p$ and $y \in \mathcal{D}(A^{i s})$

$$\|A^{-\varepsilon' + i s}x - A^{-\varepsilon + i s}x\| \leq \|A^{-\varepsilon' + i s}(x - y)\| + \|A^{-\varepsilon + i s}y - A^{-\varepsilon + i s}y\| + \|A^{-\varepsilon + i s}(y - x)\| + \|A^{-\varepsilon + i s}y - A^{-\varepsilon + i s}y\| \leq 2c\|x - y\| + \|A^{-\varepsilon' + i s}y - A^{-\varepsilon + i s}y\|.$$

Hence the sequence $\{A^{-\varepsilon + i s}x\}$ is Cauchy sequence, there is $z \in L^p$ such that $A^{-\varepsilon + i s}x \to z$ as $\varepsilon \to 0$.

Since $A^{-\varepsilon + i s}x = A^{i s}A^{-\varepsilon}x \to z$ and $A^{-\varepsilon} \to A^{-\varepsilon}$ it follows from closedness of $A$ that $x \in \mathcal{D}(A^{i s})$, and $A^{i s}x = z$. Therefore $A^{i s}$ is a bounded in view of closed graph theorem.

**Lemma 2.2.** For any $s \in \mathcal{R}$, there exist a constant $c$ such that

$$\|A^{i s}\|_p \leq ce^{s\gamma},$$

where $c$ depend on $A_p$, $p$ and $\gamma$

This Lemma follows from Lemma 2.1 and remarks before Lemma 2.1.
Stabilizability and control properties for an evolution equation

REMARK 1. For any $-\infty < s < \infty$, let $A^{is}_p$ be a bounded and $A^{is+it}_p = A^{is}_p A^{it}_p$.

If $A^{is}_p$ is strongly continuous, then there exists constants $c > 0$ and $\gamma > 0$ such that

$$\| A^{is}_p \| \leq ce^{\gamma|s|}$$

in view of properties of groups of bounded operators.

REMARK 2. In the case where $A_p$ is not invertible, R.Seely proved that

$$A^{-1}_p f = f - P_0 f, \quad f \in D(A_p)$$

$$\lim_{z \to 0} A^z_p f = f - P_0 f, \quad f \in L_p, \text{ Re } z < 0$$

where $P_0 = \frac{1}{2\pi i} \int_{|\lambda| = \epsilon} (\lambda - A_p)^{-1} d\lambda$ be the projection on the generalized null space of $A_p$ and also proved that

$$S^z = A^z_p + P_0 \text{ is a semigroup } \quad \text{and}$$

$$S^z \to I \text{ strongly as } z \to 0$$

while Lemma 2.1 is noted in terms of

$$\| S^{is}_p \| \leq ce^{i|s|}.$$

The allowable Banach spaces in this note are the $\delta$-convex space. We know that the space $X$ is $\delta$-convex if and only if the Hilbert transform is a bounded operator on $L^q(R, X), 1 < q < \infty$.

From the $\delta$-convexity of $L^p(\Omega)$ and Lemma 2.2 we obtain the following theorem by using results of G.Dore and A.Venni.

THEOREM 2.1. Let $A_p$ be an operator defined by (2.3), then for any $f \in L^r(0, T : L^p(\Omega))$ the Cauchy problem

$$\begin{cases}
  u'(t) + A_p(t) = f(t) \\
  u(0) = 0
\end{cases}$$

has a unique solution

$$u \in W^{1,r}(0, T : L^p(\Omega)) \cap L^r(0, T : W^{2,p}(\Omega) \cap W^{1,p}(\Omega)),$$

$1 < p < \infty$

and

$$u \in W^{1,r}(0, T : L^1(\Omega)) \cap L^r(0, T : W^{2,1}(\Omega) \cap W^{1,q}(\Omega)),$$

$$1 \leq q \leq \frac{n}{n-1}, \quad p = 1.$$
REMARK 3. Let $[\cdot, \cdot]_{\frac{1}{2}}$ be a complex interpolace space, then $[W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), L^p(\Omega)]_{\frac{1}{2}} = W^{1,p}_0$.  
As is seen in [11, Lemma 5.5.1], we have  
\[
W^{1,2}(0, T : L^p(\Omega)) \cap L^2(0, T : W^{2,p}(\Omega) \cap W^{1,p}(\Omega)) \\
\quad \subset C([0, T] : W^{1,p}) \\
\quad \subset C([0, T] : (D(A_p) \cap L^p(\Omega))_{\frac{1}{2}, 2})
\]
where $D(A_p) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$

3. The relation between stabilizability of solution of (3.1) and controllability

Let $U$ be complex Banach spaces. We consider the following equation 
\[
(3.1) \quad \begin{cases} 
\frac{d}{dt} u(t) = A_p u(t) + \Phi f(t) \\
x(0) = u_0
\end{cases}
\]
where $A_p$ is the operator in section 2 and $\Phi \in B(u, W^{1,p}_0(\Omega))$. We assume
\[
(3.2) \quad \sigma(A_p) \cap \{ \lambda : Re \lambda = 0 \} = \phi.
\]
Set $\sigma_+ = \sigma(A_p) \cap \{ \lambda : Re \lambda > 0 \}$, $\sigma_- = \sigma(A_p) \cap \{ \lambda : Re \lambda < 0 \}$
We assume also that
\[
(3.3) \quad \sigma_+ = \{ \lambda_1, \cdots, \lambda_N \}
\]
\[
(3.4) \quad -w_0 = \sup \{ Re \lambda : \lambda \in \sigma_- \} < 0
\]
and for each $j = 1, \cdots, N$, the spectral projection
\[
(3.5) \quad P_{\lambda_j} = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_j}} (\lambda - A_p)^{-1} d\lambda
\]
is the projection on generalized null space of $A_p$ with finite rank, where
$\Gamma_{\lambda_j}$ is a small circle centered at $\lambda_j$ such that it surrounds no point of
$\sigma(A_p)$ except $\lambda_j$.

If $A_p$ has a compact resolvent, then above assumptions is hold.

If $\lambda_j \in \sigma_+ \quad j = 1, \ldots, N$, then we have the Laurent expansion at
$\lambda = \lambda_j$ is

$$(\lambda - A)^{-1} = \sum_{n=0}^{K_{\lambda_j}} \frac{Q^n_{\lambda_j}}{(\lambda - \lambda_j)^{n+1}} + R_0(\lambda)$$

where $Q^0_{\lambda_j} = P_{\lambda_j}$, $Q_{\lambda_j} = (A - \lambda_j)P_{\lambda_j}$ and $R_0(\lambda)$ is the analytic part
of $(\lambda - A)^{-1}$ at $\lambda = \lambda_j$, hence $\lambda_j$ is a pole of $(\lambda - A)^{-1}$ whose order is
denoted by $K_{\lambda_j}$.

Moreover the order of a pole $\tilde{\lambda}$ of $(z - A^*_p)^{-1}$ is equal to $K_{\lambda}$.

The above operator $Q_{\lambda_j}$ is defined by

$$Q_{\lambda_j} = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_j}} (u - \lambda)(\lambda - A)^{-1} du$$

We have $Q^{K_{\lambda_j}}_{\lambda_j} = 0$, $Im Q_{\lambda_j} \subset Im P_{\lambda_j}$.

We put $X_{\lambda_j} = Im P_{\lambda_j}$.

Let $\Phi f \in L^2(0, \infty : W^{1,p}_0(\Omega))$ and $u(t)$ be the mild solution of
the equation (3.1) i.e.,

$$u(t) = s(t)u_0 + \int_0^t S(t-s)\Phi f(s)ds.$$  

We define the sets of attainability $R$ and the unobservable $N$ by

$$R = \{ \int_0^t S(t-s)\Phi u(s) : t \geq 0, \ u \in L^2(0, \infty : U) \} \subset W^{1,p}_0(\Omega).$$

$$N = \bigcap_{i \geq 0} Ker \Phi^* S^*(t) \subset L^p(\Omega).$$
**Definition 3.1.**

1. For any \( \lambda \in \sigma_+ \), \( S(t) = e^{tA_p} \) is \( \lambda \)-controllable if \( Cl(R) \subseteq X_\lambda \).
2. For any \( \lambda \in \sigma_+ \), \( S^*(t) = e^{tA_p^*} \) is \( \lambda \)-observable if \( N \cap X^*_\lambda = \{0\} \).

The following Lemma is proved in [7, Theorem 7.2].

**Lemma 3.1.** For any \( \lambda \in \sigma_+ \),

\[
L^p(\Omega) = \text{Ker}(\lambda - A_p)^{K\lambda} \oplus \text{Im}(\lambda - A_p)^{K\lambda},
\]

\[
X_\lambda = \text{Im}P_\lambda = \text{Ker}(\lambda - A_p)^{K\lambda},
\]

\[
L^p'(\Omega) = \text{Ker}(\bar{\lambda} - A_{p'})^{K\bar{\lambda}} \oplus \text{Im}(\bar{\lambda} - A_{p'})^{K\bar{\lambda}},
\]

\[
X^*_\lambda = \text{Im}P^*_\lambda = \text{Ker}(\bar{\lambda} - A_{p'})^{K\bar{\lambda}}.
\]

If \( \lambda \) is a pole of \((z - A)^{-1}\) then above Lemma is also hold.

**Lemma 3.2.** The following statements are equivalent;

For any \( \lambda \in \sigma_+ \),

(i) \( S(t) \) is \( \lambda \)-controllable

(ii) \( S^*(t) \) is \( \bar{\lambda} \)-observable

**Proof.** By the Hahn-Banach theorem, the necessary and sufficient condition for (i) is that \( R^\perp \subseteq X^\perp_\lambda \).

From the Lemma 3.1 and duality theorem

\[
X^\perp_\lambda = (\text{Im}P_\lambda)^\perp = \text{Ker}P^*_\lambda = \text{Im}(\bar{\lambda} - A_{p'})^{K\bar{\lambda}},
\]

\[
R^\perp = \left( \bigcup_{t \geq 0} \left\{ \int_0^t S(t-s)\Phi u(s)ds : u \in L^2(0,\infty : U) \right\} \right)^\perp
\]

\[
= \bigcap_{t \geq 0} \left\{ \int_0^t S(t-s)\Phi u(s)ds : u \in L^2(0,\infty : U) \right\}^\perp
\]

\[
= \bigcap_{t \geq 0} \text{Ker} \Phi^\perp S^\perp(t)
\]

\[
= N
\]
Hence in view of Lemma 3.1.

\[ N \cap X^*_\lambda = \text{Im}(\bar{\lambda} - A_p)^{K_\lambda} \cap X^*_\lambda = \{0\} \]

**Lemma 3.3.** The following statements are equivalent:

(i) \( S^*(t) \) is \( \bar{\lambda} \)-observable

\[ \bigcap_{j=1}^{K_\lambda-1} \text{Ker} \Phi^*(Q^*_{\lambda_j}) \cap X^*_\lambda = \{0\} \]

**Proof.** For each \( f \in X^*_\lambda \), then \( g = P^*_\lambda g \) and

\[ S^*(t)f = S^*(t)P^*_\lambda g = \frac{1}{2\pi i} \int_{\Gamma_\lambda} e^{zt}(z - A_{p'})^{-1} g dz \]

\[ = e^{\bar{\lambda}t} \frac{1}{2\pi i} \int_{\Gamma_\lambda} e^{(z-\bar{\lambda})t}(z - A_{p'})^{-1} g dz \]

\[ = e^{\bar{\lambda}t} \left\{ \sum_{n=0}^{K_\lambda-1} \frac{t^n}{n!} \left( \frac{1}{2\pi i} \int_{\Gamma_\lambda} (z - \bar{\lambda})^n(z - A_{p'})^{-1} g dz \right) \right\} \]

\[ = e^{\bar{\lambda}t} \sum_{n=0}^{K_\lambda-1} \frac{t^n}{n!} (Q^*_{\lambda_j})^n g \]

Hence, if \( f \in N \cap X^*_\lambda \), in view of above result we can see

\[ e^{\bar{\lambda}t} \sum_{n=0}^{K_\lambda-1} \frac{t^n}{n!} \Phi^*(Q^*_{\lambda_j})^n g = 0, \quad t \geq 0. \]

Therefore \( g \in (\bigcap_{j=0}^{K_\lambda-1} \text{Ker} \Phi^*(Q^*_{\lambda_j})^n) \cap (X^*_\lambda) \), here we used that

\[ e^{\bar{\lambda}t} \sum_{n=0}^{K_\lambda-1} \frac{t^n}{n!} \Phi^*(Q^*_{\lambda_j})^n g = 0, \quad t \geq 0 \]

if and only if

\[ \Phi^*(Q^*_{\lambda_j})^n g = 0, \quad n = 0, \cdots, K_\lambda - 1. \]

Now we consider the stabilizability problem for (3.1). A necessary and sufficient consider for stabilizability is given by [5, proposition 3.1].

The following theorem is the relation between the properties of controllability and stabilizability for solution of (3.1).
Theorem 3.1. The following statements are equivalent:

(i) For any \( g \in L^p(\Omega) \), there exists \( f \in L^2(0, \infty : U) \) such that the mild solution of (3.1) belongs to \( L^2(0, \infty : L^p(\Omega)) \).

(ii) \( S(t) \) is \( \lambda_j \)-controllable \( j = 1, \cdots, N \).

(iii) \( S^*(t) \) is \( \lambda_j \)-observable \( j = 1, \cdots, N \).

Proof. It follows from [5, proposition 3.1] that the necessary and sufficient condition (i) is that for \( j = 1, \cdots, N \).

\[
\{ x^* \in X^*_j : \Phi^*(A_p' - \lambda_j)Kx^* = 0, \; K = 0, \cdots, m_j - 1 \} = \{0\}.
\]

If \( x^* \in X^*_j \), then \( P_{j}^* x^* = x^* \) and

\[
\Phi^*(A_p' - \lambda_j)Kx^* = \Phi^*(A_p' - \lambda_j)K P_{j}^* x^* \leq \Phi^*(Q_{\lambda_j}^*)K x^*.
\]

By virtue of Lemma 3.3 (i) is equivalent to (iii).

Hence this theorem follows from Lemma 3.2.

4. Application for retarded system

In this section we are interested in the retared functional differential equation

(4.1)

\[
\begin{cases}
\frac{du(t)}{dt} = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s)A_2 u(t+s)ds + \Phi_0 f(t) \\
u(0) = g^0, \; u(s) = g'(s) \quad a.e \; s \in [-h,0), \; h > 0
\end{cases}
\]

where \( g = (g^0, g') \in W_0^{1,p}(\Omega) \times L^2(-h,0 : W_0^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \),
\( f \in L^2(0, \infty : U) \), \( \Phi_0 \in B(c, W_0^{1,p}(\Omega)) \), \( A_1 \in B(D(A_p), L^p(\Omega)) \) and \( A_0 = A_p \).

In particular, if \( p=1 \), then \( g \in W_0^{1,q}(\Omega) \times L^2(-h,0 : W_0^{1,2}(\Omega) \cap W_0^{1,q}(\Omega)) \) for \( 1 \leq q \leq n/n - 1 \).

Let \( Z_p = W_0^{1,p}(\Omega) \times L^2(-h,0 : D(A_p)) \) with norm

\[
\| g \|_2 = ( \| g^0 \|_{W_0^{1,p}(\Omega)} + \int_{-h}^{0} | g(s) |_{D(A_p)}^2 ds )^{\frac{1}{2}}, \; g \in Z_p.
\]
Let $S_A(t); Z_p \rightarrow M$ be the solution semigroup for (4.1) defined
$S_A(t)g = (u(t; g), u(t - 0); g))$ for any $g \in Z$ where $u(t; g)$ is the
mild solution of (4.1) (See [5]).

We can rewrite (4.1) as

$$\frac{d}{dt} x(t) = A x(t) + \Phi f(t)$$

$$x(0) = g = (g^0, g')$$

in the space $Z$, where $S_A(t) = e^{tA}$ and $\Phi$ is the operator defined by
$\Phi f = (\Phi_0 f, 0)$

In what follows we consider the operator $A$ with the assumptions
(3.2) - (3.4), but $A_p$ need not be satisfied in this case.

In the remainder of this section by $P_\lambda$ and $Q_\lambda$ we denote the operator
mentioned in section with $A_p$ replaced by $A$.

It is easily seen that the whole contents of section 3 in vailed for the
system (4.2) as in seen in [8] with this remark we have ;

**Theorem 4.1.** The following statements are equivalent;

(i) For any $g \in Z$, there exists $f \in L^2(0, \infty; U)$ such that the mild
solution $u$ of (4.1) satisfies

$$\int_0^\infty \{ |u(t)|^2_{W^1_0(i)} + \int_0^t |u(t + s)|^2_{D(A_p)} ds\} dt < \infty$$

(ii) $Ker(\lambda_j - A_p) \cap Ker\Phi^* = \{0\}, j = 1, \ldots, N.$

**Proof.** From Theorem 3.1, it follows that (i) is equivalent to the fact
that $S^*_\lambda(t)$ is $\lambda_j$-observable, $j = 1, \ldots, N$

We have only to prove that the condition (ii) is equivalent that $S^*(t)$
is $\lambda_j$-observable.

Let (ii) be hold and

$$\phi \in (\bigcap_{j=0}^{K_\lambda-1} Ker\Phi^*(Q_{\lambda_j}^*)^j) \cap Z_{\lambda_j}^*, Z_{\lambda_j}^* = ImP_{\lambda_j}^* Z$$

then $\phi \in Ker(\lambda - A_p^j)^{K_\lambda}$ and $\Phi^*(Q_{\lambda_j}^*)\phi = 0$ for $j = 0, \ldots, K_\lambda - 1$
here we used that $(Q_{\lambda_j}^*)^j = (A_p - \lambda_j)^j P_{\lambda_j}^*$ for $0 \leq j \leq K_\lambda - 1$ and
$P_{\lambda}^* \phi = 0.$
We put $\phi_1 = (\lambda_j - A_{y'})^{k_1-1}\phi$ then $\phi_1 \in \text{Ker}(\lambda_j - A_{y'})$,

$$\Phi^*\phi_1 = \Phi^*(\lambda - A_{y'})^{k_1-1}\phi = 0$$

In view of (ii) we have $\phi_1 = 0$.

Let $\phi_2 = (\lambda - A_{y'})^{k_1-1}\phi$ then $\phi_2 \in \text{Ker}(\lambda - A_{y'})$ and

$$\Phi^*\phi_2 = \Phi^*(\lambda - A_{y'})^{k_1-2}\phi = 2, \text{ hence we have } \phi_2 = 0.$$ 

Continuing this procedure, we conclude that $\phi = 0$.

Conversely, let $\phi \in \text{Ker}(\lambda_j - A_{y'}) \cap \text{Ker}\Phi^*$ and

$$\left( \bigcap_{j=0}^{k_1-1} \text{Ker}\Phi^* (Q_j^*)^j \right) \cap Z_{\lambda_j}^* \{0\} \text{ then } \phi \in Z_{\lambda_j}^* \text{ and } A_{y'}\phi = \lambda_j\phi.$$ 

Hence, since $\phi \in \text{Ker}\Phi^*$,

$$\Phi^* S^*(t)\phi = \Phi^*(e^{\lambda t}\phi) = e^{\lambda t}\Phi^*\phi = 0, \quad t \geq 0.$$ 

therefore $\phi \in \text{Ker}\Phi^* S^*(t) \cap Z_{\lambda_j}^* = \left( \bigcap_{j=0}^{k_1-1} \text{Ker}\Phi^* (Q_j^*)^j \right) \cap Z_{\lambda_j}^* = \{0\}.$

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