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ON THE STARLIKENESS BOUND OF UNIVALENT FUNCTIONS

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1. Introduction

Let S denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the unit disk $U = \{z : |z| < 1\}$. A function $f(z)$ belonging to S is said to be in the class $L(\alpha, \beta, \gamma)$ [1], [3] if and only if it satisfies

$$\left|\frac{f'(z) - 1}{\alpha f'(z) + (1-\gamma)}\right| < \beta \tag{1.2}$$

for some $\alpha(0 \leq \alpha \leq 1)$, $\beta(0 < \beta \leq 1)$, $\gamma(0 \leq \gamma < 1)$.

In particular, the class $L(1, \beta, 0) \equiv D(\beta)$ is studied by Padamanabhan [6], the class $L(0, \beta, 0) \equiv G(\beta)$ is stuied by Singh [4], [8] and the class $L(0, 1, \gamma) \equiv F(\gamma)$ is stuied by Nunokawa, Fukui, Owa, Saitoh and Sekine [5]. Let $P(\delta)$ denote the subclass of S consisting of all functions f satisfying the condition

$$\mathrm{Re}\{f'(z)\} > \delta \tag{1.3}$$

for some $\delta(0 \leq \delta < 1)$ [8].

In the present paper, we show that the starlikeness bound of functions $f(z)$ belonging to the subclass $L(\alpha, \beta, \gamma)$ of S in the unit disk, which is an improvement of the result by Nunokawa, Fukui, Owa, Saitoh and Sekine [5] when $\alpha = 0$ and $\beta = 1$. Furthermore, some considerations for starlikeness of functions with negative coefficients are shown.

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2. Boundary of Starlikeness

We begin with the following lemma.

LEMMA 1. *If a function $f(z)$ belongs to the class $L(\alpha,\beta,\gamma)$ with $\beta(2\alpha-\gamma+1)<1$, then*

$$|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2 \le |f(z)| \le |z|+\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2, \tag{2.1}$$

$$|\arg f'(z)| \le \sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{1-\alpha\beta}|z|\right\}, \tag{2.2}$$

and

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \le \sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|\right\}, \tag{2.3}$$

for $z \in U$.

Proof. From the condition (1.2) and the Schwarz Lemma, we get

$$|f'(z)-1| \le \frac{\beta(\alpha+1-\gamma)}{1-\alpha\beta}|z| \tag{2.4}$$

and

$$|f(z)-z| \le \frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2. \tag{2.5}$$

It follows that

$$|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2 \le |f(z)| \le |z|+\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2 \tag{2.6}$$

and

$$|\arg f'(z)| \le \sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{1-\alpha\beta}|z|\right\}. \tag{2.7}$$

From the condition (2.5), we have

$$\left|\arg\left(\frac{f(z)}{z}\right)\right| \le \sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|\right\}, \quad (z \in U). \tag{2.8}$$

Hence, we complete the assertion of Lemma 1.

COROLLARY 2. *If a function* $f(z)$ *belongs to the class* $D(\beta)(0 \leq \beta \leq \frac{1}{3})$, *then*

$$|z| - \frac{\beta}{1-\beta}|z|^2 \leq |f(z)| \leq |z| + \frac{\beta}{1-\beta}|z|^2, \tag{2.9}$$

and

$$|\arg f'(z)| \leq \sin^{-1}\left\{\frac{2\beta}{1-\beta}|z|\right\}, \tag{2.10}$$

for $z \in U$.

COROLLARY 3. *If a function* $f(z)$ *belongs to the class* $G(\beta)(0 < \beta \leq 1)$, *then*

$$|z| - \frac{\beta}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{\beta}{2}|z|^2, \tag{2.11}$$

and

$$|\arg f'(z)| \leq \sin^{-1}\{\beta|z|\} \tag{2.12}$$

for $z \in U$.

COROLLARY 4. *If a function* $f(z)$ *belongs to the class* $F(\gamma)(0 \leq \gamma < 1)$, *then*

$$|z| - \frac{1-\gamma}{2}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\gamma}{2}|z|^2, \tag{2.13}$$

and

$$|\arg f'(z)| \leq \sin^{-1}\{(1-\gamma)|z|\} \tag{2.14}$$

for $z \in U$.

From the above Lemma, we derive

THEOREM 5. *If a function $f(z)$ is in the class $L(\alpha,\beta,\gamma)$ with $\beta(2\alpha-\gamma+1)<1$, then $f(z)$ is starlike in $|z|<r_0<1$, where r_0 is the root of the equation*

(2.15)
$$\log\left(\frac{1-\left\{\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}\right\}^2\left\{|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2\right\}^2}{1-|z|^2}\right)$$
$$+\sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{1-\alpha\beta}|z|\right\}=\pi.$$

Proof. By Lemma 1, for $f(z)$ in the class $L(\alpha,\beta,\gamma)$, we get

$$|f(z)|<1+\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z| \quad (z\in U). \tag{2.16}$$

By using Loewner's differential equation and the same manner in [2], we define the function $g(z)$ by

$$f(z)=e^{t_0}g(z)\equiv\frac{2-\beta(\alpha-1+\gamma)}{2(1-\alpha\beta)}g(z). \tag{2.17}$$

Then $g(z)$ ia analytic in U, and satisfies $g(0)=0$ and $|g(z)|<1$ for $z\in U$. Thus, by the Schwarz Lemma, we get $|g(z)|\le|z|$ for $z\in U$. From (2.1) and (2.17), we have

(2.18)
$$\begin{aligned}|g(z)|&=\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}|f(z)|\\&\ge\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}\left\{|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2\right\}.\end{aligned}$$

Hence, we get

(2.19)
$$\left|\arg\left(\frac{z^2f'(z)}{f(z)^2}\right)\right|=\left|\arg\left(\frac{z^2g'(z)}{g(z)^2}\right)\right|$$

$$= \left| \int_{\arg[1/z^2]}^{\arg[g'(z)/g(z)^2]} d \arg \left(\frac{g'(z)}{g(z)^2} \right) \right|$$

$$\leq \int_{|z|}^{|g(z)|} \frac{-2|g(z)|}{1-|g(z)|^2} d\,|g(z)|$$

$$\leq \int_{|z|}^{\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}\left\{|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2\right\}} \frac{-2|g(z)|}{1-|g(z)|^2} d\,|g(z)|$$

$$= \log \left(\frac{1-\left\{\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}\right\}^2 \left\{|z|-\frac{\beta(\alpha+1-\gamma)}{2(1-\alpha\beta)}|z|^2\right\}^2}{1-|z|^2} \right)$$

for $0 < |z| < 1$. From (2.7) and (2.19), we get

(2.20)

$$\left|2\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \left|\arg\left(\frac{z^2 f'(z)}{f(z)^2}\right)\right| + |\arg(f'(z))|$$

$$\leq \log \left(\frac{1-\left\{\frac{2(1-\alpha\beta)}{2-\beta(\alpha-1+\gamma)}\right\}^2 \left\{|z|-\frac{\beta(\alpha+1-\gamma)}{2(1+\alpha\beta)}|z|^2\right\}^2}{1-|z|^2} \right)$$

$$+ \sin^{-1}\left\{\frac{\beta(\alpha+1-\gamma)}{1-\alpha\beta}|z|\right\} < \pi.$$

for some $|z| < r_0 < 1$. This completes the proof of Theorem.

COROLLARY 6. *If a function $f(z)$ is in the class $D(\beta)(0 < \beta \leq \frac{1}{3})$, then $f(z)$ is starlike in $|z| < r_0 < 1$, where r_0 is the root of the equation*

$$\text{(2.21)}\quad \log \left(\frac{1-(1-\beta)^2\left\{|z|-\frac{\beta}{1-\beta}|z|^2\right\}^2}{1-|z|^2} \right) + \sin^{-1}\left\{\frac{2\beta}{1-\beta}|z|\right\} = \pi.$$

COROLLARY 7. *If a function $f(z)$ is in the class $G(\beta)(0 < \beta \leq 1)$, then $f(z)$ is starlike in $|z| < r_0 < 1$, where r_0 is the root of the equation*

$$\text{(2.22)}\qquad \log \left(\frac{1-\left\{\frac{2}{(2+\beta)^2}\right\}\left\{|z|-\frac{\beta}{2}|z|^2\right\}^2}{1-|z|^2} \right) + \sin^{-1}\{\beta|z|\} = \pi.$$

COROLLARY 8. *If a function $f(z)$ is in the class $F(\gamma)(0 \leq \gamma < 1)$, then $f(z)$ is starlike in $|z| < r_0 < 1$, where r_0 is the root of the equation*

$$\text{(2.23)} \quad \log\left(\frac{1-\left\{\frac{2}{3-\gamma}\right\}^2\left\{|z|-\frac{1-\gamma}{2}|z|^2\right\}^2}{1-|z|^2}\right)+\sin^{-1}\{(1-\gamma)|z|\}=\pi.$$

3. Starlikeness of Functions with Negative Coefficients.

Denoting by T the subclass of S consisting of functions of the form

$$\text{(3.1)} \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

We define the class $T^*(\delta)$ and $L^*(\alpha,\beta,\gamma)$ by

$$T^*(\delta) = T \cap S^*(\delta) \quad (0 \leq \delta < 1), \qquad L^*(\alpha,\beta,\gamma) = T \cap L(\alpha,\beta,\gamma),$$

where $S^*(\delta)$ is the subclass of S consisting of all atarlike functions of order δ.

In order to give our result, we have to recall here the following lemma due to Silverman [7].

LEMMA 9. *A function $f(z)$ defined by (3.1) is in the class $T^*(\delta)(0 \leq \delta < 1)$ if and only if*

$$\text{(3.2)} \qquad \sum_{n=2}^{\infty}(n-\delta)a_n \leq 1-\delta.$$

LEMMA 10. [1],[3]. *A function $f(z)$ defined by (3.1) is in the class $L^*(\alpha,\beta,\gamma)$ if and only if*

$$\text{(3.3)} \qquad \sum_{n=2}^{\infty}(1+\alpha\beta)n|a_n| \leq \beta(\alpha+1-\gamma).$$

THEOREM 11. *A function $f(z)$ defined by* (3.1) *is in the class $L^*(\alpha, \beta, \gamma)$, then $f(z)$ is in the class $T^*(\delta)$, where*

$$\delta = \frac{1+\beta\gamma-\beta}{1+\alpha\beta}.$$

Proof. From Lemma 10, we note that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{\beta(\alpha+1-\gamma)}{1+\alpha\beta}, \tag{3.4}$$

and from Lemma 9, we get

$$\sum_{n=2}^{\infty}(n-\delta)|a_n| \leq 1-\delta. \tag{3.5}$$

Hence, $f(z)$ is in the class $T^*(\delta)$ for

$$\delta \leq \frac{1+\beta\gamma-\beta}{1+\alpha\beta}.$$

The author have proved the same result for $L(\alpha,\beta,\gamma)$ in [3].

COROLLARY 12. [5]. *A function $f(z)$ is in the class $T \cap F(\gamma)(0 \leq \gamma < 1)$, then $f(z)$ belongs to the class $T^*(\gamma)$, that is, $f(z)$ is starlike of order γ in U.*

Proof. Since $F(\gamma) \equiv L(0,1,\gamma)$, $f(z)$ is in the class $T^*(\delta)$ for $\delta \leq \gamma$.

In [1], Kim and Lee proved that $P^*(\delta) = L^*(\delta, \frac{1-\delta}{1+\delta^2}, 0)$ for some $\delta(0 \leq \delta < 1)$. From this fact, we have the following corollary.

COROAALRY 13. [5]. *If a function $f(z) \in T$ satisfies* $\mathrm{Re}[f'(z)] > \delta$ $(0 \leq \delta < 1)$, *then $f(z) \in T^*(\delta)$, that is $f(z)$ is starlike of order δ in U.*

Proof. Assume that $f(z)$ is in the class $P^*(\delta)$. By Theorem 11, we have

$$f(z) \in L^*\left(\delta, \frac{1-\delta}{1+\delta^2}, 0\right) = T^*\left(\frac{1-\frac{1-\delta}{1+\delta^2}}{1+\frac{1-\delta}{1+\delta^2}}\right) = T^*(\delta).$$

COROAALRY 14. [5]. *If a function $f(z)$ defined by (3.1) is close-to-convex in U, then $f(z)$ is starlike in U.*

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