Pusan Kyŏngnam Math. J. 7(1991), No. 2, pp 95–108

COMMON FIXED POINTS OF WEAKLY* COMMUTING MAPPINGS

Y. J. Cho, S. M. Kang and M. S. Khan

1. Introduction and Preliminaries

The concept of 2-metric spaces has been investigated initially by Gähler in a series of papers [1]–[3] and has been developed extensively by Gähler and many others.

A 2-metric is a set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions:

(M_1) For two distinct points x, y in X, there exists a point z in X such that $d(x, y, z) \neq 0$,
(M_2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,
(M_3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,
(M_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, u in X.

Then d is called a 2-metric for the space X and (X, d) is called a 2-metric space. It has been shown by Gähler [1] that a 2-metric d is non-negative and although d is a continuous function of any one of its three arguments, it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its argument will be called continuous.

On the other hand, a number of mathematicans ([4]–[15], [17], [19]–[29]) have studied the aspects of fixed point theory in the setting of the 2-metric spaces. They have been motivated by various concepts already known for ordinary metric spaces and have thus introduced analogues of various concepts in the frame work of the 2-metric spaces Especially, Khan [7] and Naidu–Prasad [17] introduced the concept of weakly commuting pairs of self-mappings on a 2-metric space and the notion of weak continuity of a 2-metric, respectively, and they

Received October 20, 1991.

have proved several common fixed point theorems by using the weakly commuting pairs of self-mappings on a 2-metric space and the weak continuity of a 2-metric.

In this paper, we give some common fixed point theorems for weakly* commuting mappings from a complete 2-metric space and some theorems on the convergence of self-mappings on a complete 2-metric space and the existence of their fixed points. Our main theorems extend the Banach Contraction Principle in 2-metric spaces

Now, we shall give some definitions:

DEFINITION 1.1. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be *convergent* to a point x in X if $\lim_{n\to\infty} d(x_n, x, z) = 0$ for all z in X. Then x is called the *limit* of the sequence $\{x_n\}$ in X.

DEFINITION 1.2. A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a *Cauchy sequence* if $\lim_{m,n\to\infty} d(x_m, x_n, z) = 0$ for all z in X.

DEFINITION 1.3. A 2-metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Note that in a 2-metric space a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X ([17]).

DEFINITION 1.4. Let S and T be two mappings from a 2-metric space (X, d) into itself. Then a pair (S, T) is said to be *weakly commuting* on X if $d(STx, TSx, z) \leq d(Tx, Sx, z)$ for all x, z in X.

DEFINITION 1.5. Let S and T be two mappings from a 2-metric space (X, d) is said to be *weakly* commuting* on X if $d(STx, TSx, z) \leq d(S^2x, T^2x, z)$ for all x, z in X.

Note that, if $S^2 = S$ and $T^2 = T$ in Definition 1.5, then a weakly* commuting pair (S, T) is weakly commuting on X. A commuting pair (S, T) on X is also weakly commuting and weakly* commuting, but the converses are not true.

DEFINITION 1.6. A mapping S from a 2-metric space (X, d) into itself is said to be *sequentially continuous* at x if for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, x, z) = 0$ for all z in X, $\lim_{n\to\infty} d(Sx_n, Sx, z) = 0$.

DEFINITION 1.7. Let S and T be two mappings from a 2-metric space (X,d) into itself. Then a sequence $\{x_n\}$ in X is said to be *asymptotically S^2-regular* with respect to T^2 if $\lim_{n\to\infty} d(S^2x_n, T^2x_n, z) = 0$ for all z in X.

Throughout this paper, let (X,d) be a 2-metric space with the continuous 2-metric d. Let $\mathbf{N}$ and $\mathbf{R}^+$ be the set all natural numbers and non-negative real numbers, respectively, and $\mathcal{F}$ the family of mappings ϕ from $(\mathbf{R}^+)^5$ into $\mathbf{R}^+$ such that ϕ is upper semicontinuous and nondecreasing in each coordinate variable, for any $t > 0$,

$$\phi(t,t,0,\alpha t,0) \le \beta t, \qquad \text{and } \phi(t,t,0,0,\alpha t) \le \beta t,$$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$,

$$\gamma(t) = \phi(t,t,a_1t,a_2t,a_3t) < t,$$

where $\gamma : \mathbf{R}^+ \to \mathbf{R}^+$ is a mapping and $a_1 + a_2 + a_3 = 4$.

For our main theorems, we need the following lemmas:

LEMMA 1.1 ([16]). *For any $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the n-times composition of γ.*

Let A, S and T be mappings from a 2-metric space (X,d) into itself such that

$$A(X) \subset S(X) \cap T(X), \tag{1.1}$$

(1.2)
$$\begin{aligned} d^2(Ax,Ay,z) \le \phi(d^2(Sx,Ty,z),\, & d(Sx,Ax,z)\cdot d(Ty,Ay,z),\\ & d(Sx,Ay,z)\cdot d(Ty,Ax,z),\\ & d(Sx,Ax,z)\cdot d(Ty,Ax,z),\\ & d(Sx,Ay,z)\cdot d(Ty,Ay,z)), \end{aligned}$$

where $\phi \in \mathcal{F}$.

Then, by (1.1) since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $A(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Ax_1 = Sx_2$ and so on.

Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n+1} = Sx_{2n+2} = Ax_{2n+1} \tag{1.3}$$

for $n = 0,1,2,\cdots$.

LEMMA 1.2. *Let A, S and T be mappings from a 2-metric space (X, d) into itself satisfying the conditions (1.1) and (1.2). Then we have the following:*
(1) for every $n \in \mathbf{N}_0$, $d(y_n, y_{n+1}, y_{n+2}) = 0$,
(2) for every $i, j, k \in \mathbf{N}_0$, $d(y_i, y_j, y_k) = 0$, where $\{y_n\}$ is the sequence in X defined by (1.3).

Proof. (1) In (1.2), taking $x = x_{2n+2}$, $y = x_{2n+1}$ and $z = y_{2n}$, we have

$$\begin{aligned} d^2(y_{2n+2}, y_{2n+1}, y_{2n}) &= d^2(Ax_{2n+2}, Ax_{2n+1}, y_{2n}) \\ &\leq \phi(d^2(y_{2n+1}, y_{2n}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n+2}, y_{2n}) \cdot d(y_{2n}, y_{2n+1}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n+1}, y_{2n}) \cdot d(y_{2n}, y_{2n+2}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n+2}, y_{2n}) \cdot d(y_{2n}, y_{2n+2}, y_{2n}), \\ &\qquad d(y_{2n+1}, y_{2n+1}, y_{2n}) \cdot d(y_{2n}, y_{2n+1}, y_{2n})) \\ &= \phi(0,0,0,0,0) \\ &< 0 \end{aligned}$$

and so $d(y_{2n+2}, y_{2n+1}, y_{2n}) = 0$. Similarly, we have

$$d(y_{2n+1}, y_{2n+2}, y_{2n+3}) = 0.$$

Hence, $d(y_n,\ y_{n+1},\ y_{n+2}) = 0$ for every $n \in \mathbf{N}_0$.

(2) For all $z \in X$, let $d_n(z) = d(y_n, y_{n+1}, z)$, $n = 0, 1, 2, \cdots$. By (1), we have

$$\begin{aligned} d(y_n, y_{n+2}, z) &\leq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z) \\ &= d(y_n, y_{n+1}, z) + d(y_{n+1}, y_{n+2}, z) \\ &= d_n(z) + d_{n+1}(z). \end{aligned}$$

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (1.2), we have

$$\begin{aligned} d^2_{2n+1}(z) &= d^2(y_{2n+2}, y_{2n+1}, z) \\ &= d^2(Ax_{2n+2}, Ax_{2n+1}, z) \end{aligned}$$

$$
\begin{aligned}
&\le \phi(d^2(y_{2n+1}, y_{2n}, z), d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+1}, z),\\
&\quad d(y_{2n+1}, y_{2n+1}, z) \cdot d(y_{2n}, y_{2n+2}, z),\\
&\quad d(y_{2n+1}, y_{2n+2}, z) \cdot d(y_{2n}, y_{2n+2}, z),\\
&\quad d(y_{2n+1}, y_{2n+1}, z) \cdot d(y_{2n}, y_{2n+1}, z))\\
&= \phi(d_{2n}^2(z), d_{2n+1}(z) \cdot d(_{2n}(z), 0,\\
&\quad d_{2n+1}(z)(d_{2n}(z) + d_{2n+1}(z)), 0).
\end{aligned}
$$

Now, we shall prove that $\{d_n(z)\}$ is a non-increasing sequence in $\mathbf{R}^+$.

In fact, suppose that $d_{n+1}(z) > d_n(z)$ for some $n \in N$. Then, for some $\alpha < 2$, $d_{n+1}(z) + d_n(z) = \alpha d_{n+1}(z)$. Since ϕ is non-decreasing in each variable and $\beta < 1$ for some $\alpha < 2$, by(1.2), we have

$$
\begin{aligned}
d_{2n+1}^2(z) &\le \phi(d_{2n+1}^2(z), d_{2n+1}^2(z), 0, \alpha d_{2n+1}^2(z), 0)\\
&\le \beta d_{2n+1}^2(z)\\
&< d_{2n+1}^2(z)
\end{aligned}
$$

and

$$
\begin{aligned}
d_{2n+2}^2(z) &\le \phi(d_{2n+2}^2(z), d_{2n+2}^2, 0, 0, \alpha d_{2n+1}^2(z))\\
&\le \beta d_{2n+2}^2(z)\\
&< d_{2n+2}^2(z).
\end{aligned}
$$

Hence, for every $n \in \mathbf{N}_0$, $d_n^2(z) \le \beta d_n^2(z) < d_n^2(z)$, which is a contradiction. Therefore, $\{d_n(z)\}$ is a a non-increasing sequence in $\mathbf{R}^+$.
By using the fact that the sequence $\{d_n(z)\}$ is non-increasing, we have the following:

A. $d_0(y_0) = 0 \implies d_n(y_0) = 0$ for every $n \in \mathbf{N}$,

B. $d_{m-1}(y_m) = 0$ for any $n \in \mathbf{N} \implies d_n(y_m) = 0$ for all $n \ge m-1$,

C. $d_{m-1} = 0 = d_{m-1}(y_n)$ for $0 \le n < m-1$ and (M_4)
$$\implies d_n(y_m) \le d_n(y_{m-1}) \le d_n(y_{m-2}) \le \cdots,$$

D. $d_n(y_{n+1}) = 0 \implies d_n(y_m) = 0$ for $0 \le n < m-1$.

Thus, we have shown $d_n(y_m) = 0$ for all $m, n = 0, 1, 2, \cdots$.

E. $d_{j-1}(y_j) = 0 = d_{j-1}(y_k)$ for any $i, j, k \in \mathbf{N}_0$ with $i < j$
$$\implies d(y_i, y_j, y_k) \le d(y_i, y_{j-1}, y_k).$$

Therefore, by using the above inequality in E, repeatedly, we have

$$d(y_i, y_j, y_k) \le d(y_i, y_i, y_k) = 0,$$

which means that $d(y_i, y_j, y_k) = 0$ for every $i, j, k \in \mathbf{N}_0$.

LEMMA 1.3. *Let A, S and T be mappings from a 2-metric space (X, d) into itself satisfying the conditions* (1.1) *and* (1.2). *Then the sequence $\{y_n\}$ defined by* (1.3) *is a Cauchy sequence in X.*

Proof. In the proof of Lemma 1.2, since $\{d_n(z)\}$ is a non-decreasing sequence in $\mathbf{R}^+$, by (1.2), we have

$$\begin{aligned}
d_1^2(z) &= d^2(y_1, y_2, z) \\
&= d^2(Ax_1, Ax_2, z) \\
&\leq \phi(d_0^2(z), d_0(z)\cdot d_1(z), 0, 0, (d_0(z)+d_1(z))d_1(z)) \\
&\leq \phi(d_0^2(z), d_0^2(z), d_0^2(z), d_0^2(z), 2d_0^2(z)) \\
&= \gamma(d_0^2(z)).
\end{aligned}$$

In general, we have $d_n^2(z) \leq \gamma^n(d_0^2(z))$, which implies that, if $d_0(z) > 0$, by Lemma 1.1,

$$\lim_{n\to\infty} d_n^2(z) \leq \lim_{n\to\infty} \gamma^n(d_0^2(z)) = 0.$$

Therefore, we have $\lim_{n\to\infty} d_n(z)) = 0$. For $d_0(z) = 0$, since $\{d_n(z)\}$ is non-increasing, we have clearly $\lim_{n\to\infty} d_n(z) = 0$.

Now, we shall prove that $\{y_n\}$ is a Cauchy sequence in X. Since $\lim_{n\to\infty} d_n(z) = 0$, it is sufficient to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is a Cauchy sequence in X. Suppose that the sequence $\{y_{2n}\}$ is not a Cauchy sequence in X. Then there exist a point $a \in X$, an $\epsilon > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $k \leq n_k < m_k$,

$$(1.4) \qquad d(y_{2n_k}, y_{2m_k}, a) \geq \epsilon \quad \text{and} \quad d(y_{2n_k}, y_{2m_k-2}, a) < \epsilon$$

for all $k = 1, 2, \cdots$. By Lemma 1.2 and (M_4), we have

$$\begin{aligned}
d(y_{2n_k}, y_{2m_k}, a) - d(y_{2n_k}, y_{2m_k-2}, a) &\leq d(y_{2m_k-2}, y_{2m_k}, a) \\
&\leq d_{2m_k-2}(a) + d_{2m_k-1}(a).
\end{aligned}$$

Since $\{d(y_{2n_k}, y_{2m_k}, a) - \epsilon\}$ and $\{\epsilon - d(y_{2n_k}, y_{2m_k-2}, a)\}$ are sequences in $\mathbf{R}^+$ and $\lim_{i\to\infty} d_i(a) = 0$, we have

$$(1.5) \quad \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}, a) = \epsilon \quad \text{and} \quad \lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-2}, a) = \epsilon.$$

Note that, by (M_4)

$$|d(x,y,a) - d(x,y,a)| \le d(a,b,x) + d(a,b,y) \tag{1.6}$$

for all $x, y, a, b \in X$.

Taking $x = y_{2n_k}$, $y = a$, $a = y_{2m_k - 1}$ and $b = y_{2m_k}$ in (1.6) and using Lemma 1.2 and (1.5), we get

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}, a) = \epsilon. \tag{1.7}$$

Once again, by using Lemma 1.2, (1.5) and (1.6), we get

$$\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k}, a) = \epsilon \text{ and } \lim_{k\to\infty} d(y_{2n_k-1}, y_{2m_k-1}, a) = \epsilon. \tag{1.8}$$

By (1.2), we have

(1.9)

$$\begin{aligned} d^2(y_{2m_k}, y_{2n_k+1}, a) = d^2(Ax_{2m_k}, Ax_{2n_k+1}, a) \\ &\le \phi(d^2(y_{2m_k-1}, y_{2n_k}, a), \\ &\quad d(y_{2m_k-1}, y_{2m_k}, a) \cdot d(y_{2n_k}, y_{2n_k+1}, a), \\ &\quad d(y_{2m_k-1}, y_{2n_k+1}, a) \cdot d(y_{2n_k}, y_{2m_k}, a), \\ &\quad d(y_{2m_k-1}, y_{2m_k}, a) \cdot d(y_{2n_k}, y_{2m_k}, a), \\ &\quad d(y_{2m_k-1}, y_{2n_k+1}, a) \cdot d(y_{2n_k}, y_{2n_k+1}, a)). \end{aligned}$$

Using (1.4), (1.5), (1.6) and (1.7), since $\phi \in \mathcal{F}$, we have

$$\epsilon^2 \le \phi(\epsilon^2, 0, \epsilon^2, 0, 0) \le \gamma(\epsilon^2) < \epsilon^2$$

as $k \to \infty$ in (1.9), which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X.

2. A Common Fixed Point Theorem.

By using Lemma 1.3, we have the following:

THEOREM 2.1. *Let A, S and T be mappings from a complete 2-metric space (X,d) into itself satisfying* (1.1), (1.2) *and the following conditions:*
(2.1) S and T are sequentially continuous,
(2.2) the pairs (A,S) and (A,T) are weakly commuting on X,*
(2.3) there exists a sequence which is asymptotically A^2-regular with respect to S^2 and T^2.
Then A, S and T have a unique common fixed point in X.

Proof. By Lemma 1.3, since the sequence $\{y_n\}$ defined by (1.3) is a Cauchy sequence in X and (X,d) is a complete 2-metric space, it converges to some point u in X. Note that subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ of $\{y_n\}$ also converges to u. Since S and T are sequentially continuous,

$$Sy_{2n} = STx_{2n+1} \to Su \text{ and } Ty_{2n+1} = TSx_{2n+2} \to Tu \text{ as } n \to \infty.$$

By (M_4) and (1.2), we have

$$\begin{aligned}
&d(STx_{2n+1}, TSx_{2n+2}, z)\\
&\quad = d(SAx_{2n}, TAx_{2n+1}, z)\\
&\quad \le d(SAx_{2n}, TAx_{2n+1}, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z)\\
&\qquad + d(ASx_{2n}, TAx_{2n+1}, z)\\
&\quad \le d(SAx_{2n}, TAx_{2n+1}, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z)\\
&\qquad + d(ASx_{2n}, TAx_{2n+1}, ATx_{2n+1}) + d(ATx_{2n+1}, TAx_{2n+1}, z)\\
&\qquad + d(ASx_{2n}, ATx_{2n+1}, z)\\
&\quad \le d(SAx_{2n}, TAx_{2n+1}, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z)\\
&\qquad + d(ASx_{2n}, TAx_{2n+1}, ATx_{2n+1}) + d(ATx_{2n+1}, TAx_{2n+1}, z)\\
&\qquad + [\phi(d^2(S^2x_{2n}, T^2x_{2n+1}, z),\\
&\qquad d(S^2x_{2n}, ASx_{2n}, z)\cdot d(T^2x_{2n+1}, ATx_{2n+1}, z),\\
&\qquad d(S^2x_{2n}, ATx_{2n+1}, z)\cdot d(T^2x_{2n+1}, ASx_{2n}, z),\\
&\qquad d(S^2x_{2n}, ASx_{2n}, z)\cdot d(T^2x_{2n+1}, ASx_{2n}, z),\\
&\qquad d(S^2x_{2n}, ATx_{2n+1}, z)\cdot d(T^2x_{2n+1}, ATx_{2n+1}, z))]^{\frac{1}{2}}\\
&\quad \le d(SAx_{2n}, TAx_{2n+1}, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z)
\end{aligned}$$

$$
\begin{aligned}
&+ d(ASx_{2n}, TAx_{2n+1}, ATx_{2n+1}) + d(ATx_{2n+1}, TAx_{2n+1}, z) \\
&+ [\phi(d^2(S^2x_{2n}, T^2x_{2n+1}, z), \\
&\{d(S^2x_{2n}, ASx_{2n}, SAx_{2n} + d(S^2x_{2n}, SAx_{2n}, z) \\
&+ d(SAx_{2n}, ASx_{2n}, z)\} \cdot \{d(T^2x_{2n+1}, ATx_{2n+1}, TAx_{2n+1}) \\
&+ d(T^2x_{2n+1}, TAx_{2n+1}, z) + d(TAx_{2n+1}, ATx_{2n+1}, z)\}, \\
&\{d(S^2x_{2n}, ATx_{2n+1}, TAx_{2n+1}) + d(S^2x_{2n}, TAx_{2n+1}, z) \\
&+ d(TAx_{2n+1}, ATx_{2n+1}, z)\} \cdot \{d(T^2x_{2n+1}, ASx_{2n}) \\
&+ d(T^2x_{2n+1}, SAx_{2n}, z) + d(SAx_{2n}, ASx_{2n}, z)\}, \\
&\{d(S^2x_{2n}, ASx_{2n}, SAx_{2n}) + d(S^2x_{2n}, SAx_{2n}, z) \\
&+ d(SAx_{2n}, ASx_{2n}, z)\} \cdot \{d(T^2x_{2n+1}, ASx_{2n}, SAx_{2n}) \\
&+ d(T^2x_{2n+1}, SAx_{2n}, z) + d(SAx_{2n}, ASx_{2n}, z)\}, \\
&\{d(S^2x_{2n}, ATx_{2n+1}, TAx_{2n+1}) + d(S^2x_{2n}, TAx_{2n+1}, z) \\
&+ d(TAx_{2n+1}, ATx_{2n+1}, z)\} \cdot \{d(T^2x_{2n+1}, ATx_{2n+1}, TAx_{2n+1}) \\
&+ d(T^2x_{2n+1}, TAx_{2n+1}, z) + d(TAx_{2n+1}, ATx_{2n+1}, z)\})]^{\frac{1}{2}}.
\end{aligned}
$$

Letting $n \to \infty$ and $d(Su, Tu, z) > 0$, we have

$$
\begin{aligned}
d(Su, Tu, z) &\leq [\phi(d^2(Su, Tu, z), 0, d^2(Su, Tu, z), 0, 0)]^{\frac{1}{2}} \\
&\leq [\gamma(d^2(Su, Tu, z))]^{\frac{1}{2}} \\
&< d(Su, Tu, z),
\end{aligned}
$$

which is a contradiction. Therefore, $Su = Tu$.

Next, we shall show that $Au = Su$. By (M_4) and (1.2), we have

$$
\begin{aligned}
d(SAx_{2n}, Au, z) &\leq d(SAx_{2n}, Au, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z) \\
&\quad + d(ASx_{2n}, Au, z) \\
&\leq d(SAx_{2n}, Au, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z) \\
&\quad + [\phi(d^2(S^2x_{2n}, Tu, z), \\
&\quad d(S^2x_{2n}, ASx_{2n}, z) \cdot d(Tu, Au, z), \\
&\quad d(S^2x_{2n}, Au, z) \cdot d(Tu, ASx_{2n}, z),
\end{aligned}
$$

$$\begin{aligned}
& d(S^2x_{2n}, ASx_{2n}, z) \cdot d(Tu, ASx_{2n}, z), \\
& d(S^2x_{2n}, Au, z) \cdot d(Tu, Au, z))]^{\frac{1}{2}} \\
\leq\ & d(SAx_{2n}, Au, ASx_{2n}) + d(SAx_{2n}, ASx_{2n}, z) \\
& + [\phi(d^2(S^2x_{2n}, Tu, z), \\
& \{d(S^2x_{2n}, ASx_{2n}, SAx_{2n}) + d(S^2x_{2n}, SAx_{2n}, z) \\
& + d(SAx_{2n}, ASx_{2n}, z)\} \cdot d(Tu, Au, z), \\
& d(S^2x_{2n}, Au, z) \cdot \{d(Tu, ASx_{2n}, SAx_{2n}, z) \\
& + d(Tu, SAx_{2n}, z) + d(SAx_{2n}, ASx_{2n}, z)\}, \\
& \{d(S^2x_{2n}, ASx_{2n}, SAx_{2n}) + d(S^2x_{2n}, SAx_{2n}, z) \\
& + d(SAx_{2n}, ASx_{2n}, z)\} \cdot \{d(Tu, ASx_{2n}, SAx_{2n}) \\
& + d(Tu, SAx_{2n}, z) + d(SAx_{2n}, ASx_{2n}, z)\}, \\
& d(S^2x_{2n}, Au, z) \cdot d(Tu, Au, z))]^{\frac{1}{2}}.
\end{aligned}$$

Letting $n \to \infty$ and $d(Su, Au, z) > 0$, we have

$$\begin{aligned}
d(Su, Au, z) &\leq [\phi(0, 0, 0, 0, d^2(Su, Au, z))]^{\frac{1}{2}} \\
&\leq [\gamma(d^2(Su, Au, z))]^{\frac{1}{2}} \\
&< d(Su, Au, z),
\end{aligned}$$

which is a contradiction. Therefore, $Au = Su = Tu$.

Finally, in order to prove that u is a common fixed point of A, S and T, using (1.2) again, we have

$$\begin{aligned}
d(Au, Ax_{2n+1}, z) \leq & [\phi(d^2(Su, Tx_{2n+1}, z), \\
& d(Su, Au, z) \cdot d(Tx_{2n+1}, Ax_{2n+1}, z), \\
& d(Su, Ax_{2n+1}, z) \cdot d(Tx_{2n+1}, Au, z), \\
& d(Su, Au, z) \cdot d(Tx_{2n+1}, Au, z), \\
& d(Su, Ax_{2n+1}, z) \cdot d(Tx_{2n+1}, Ax_{2n+1}, z))]^{\frac{1}{2}}.
\end{aligned}$$

Letting $n \to \infty$ and $d(Au, u, z) > 0$, we have

$$\begin{aligned}
d(Au, u, z) &\leq [\phi(d^2(Au, u, z), 0, d^2(Au, u, z), 0, 0)]^{\frac{1}{2}} \\
&\leq [\gamma(d^2(Au, u, z))]^{\frac{1}{2}} \\
&< d(Au, u, z),
\end{aligned}$$

which is a contradiction. Therefore, $Au = Su = Tu = u$, that is, u is a common fixed point of A, S and T.

The uniqueness of the common fixed point u follows easily. In fact, let u_1 and u_2 be common fixed points of A, S and T. By (1.2),

$$\begin{aligned} d(u_1, u_2, z) &= d(Au_1, Au_2, z) \\ &\leq [\phi(d^2(Su_1, Tu_2, z), d(Su_1, Au_1, z) \cdot d(Tu_2, Au_2, z), \\ &\quad d(Su_1, Au_2, z) \cdot d(Tu_2, Au_1, z), \\ &\quad d(Su_1, Au_1, z) \cdot d(Tu_2, Au_1, z), \\ &\quad d(Su_1, Au_2, z) \cdot d(Tu_2, Au_2, z))]^{\frac{1}{2}} \\ &\leq [\phi(d^2(u_1, u_2, z), 0, d^2(u_1, u_2, z), 0, 0)]^{\frac{1}{2}} \\ &\leq [\gamma(d^2(u_1, u_2, z))]^{\frac{1}{2}} \\ &< d^2(u_1, u_2, z), \end{aligned}$$

which is a contradiction. Therefore, u is a unique common fixed point of A, S and T.

REMARK. (1) Theorem 2.1 is an extension of Theorem 2.5 [18] to 2-metric spaces.

(2) In Theorem 2.1, if u is a common fixed point of A, S and T, then A is sequentially continuous at u.

3. Convergence of Self-mappings on a 2-metric Space and Their Fixed Points

In this section, we give two theorems on the convergence of self-mappings from a 2-metric space into itself and the existence of their fixed points. The following theorems follows easily from Theorem 2.3.

THEOREM 3.1. *Let $\{A_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a complete 2-metric space (X, d) into itself such that*
(3.1) the sequences $\{A_n\}$, $\{S_n\}$ and $\{T_n\}$ converge uniformly to self-mappings A, S and T on X, respectively,
(3.2) S and T are sequentially continuous.
Suppose that, for $n = 1, 2, \cdots$, x_n is a common fixed point of A_n and S_n, and y_n is a common fixed point of A_n and T_n.

Further, let self-mappings A, S and T on X satisfy the conditions (1.1), (1.2), and (2.1). If x is a common fixed point of A, S and T and $\sup\{d(x_n, x, z)\} < \infty$ *and* $\sup\{d(y_n, x, z)\} < \infty$ *for all* $z \in X$, *then* $x_n \to x$ *and* $y_n \to x$ *as* $n \to \infty$.

THEOREM 3.2. *Let $\{A_n\}$, $\{S_n\}$ and $\{T_n\}$ be sequences of mappings from a complete 2-metric space (X, d) into itself such that, for $n = 1, 2, \cdots$,*
(3.3) $A_n(X) \subset S_n(X) \cap T_n(X)$,
(3.4) the pairs (A_n, S_n) and (A_n, T_n) are *weakly* commuting on* X,

(3.5)
$$\begin{aligned} d^2(A_nx, A_ny, z) \le \phi(d^2(S_nx, T_ny, z), &\\ &d(S_nx, A_nx, z) \cdot d(T_ny, A_ny, z), \\ &d(S_nx, A_ny, z) \cdot d(T_ny, A_nx, z), \\ &d(S_nx, A_nx, z) \cdot d(T_ny, A_nx, z), \\ &d(S_nx, A_ny, z) \cdot d(T_ny, A_ny, z)) \end{aligned}$$

for all $x, y, z \in X$, where $\phi \in \mathcal{F}$.
If the sequences $\{A_n\}$, $\{S_n\}$ and $\{T_n\}$ converges uniformly to self-mappings A, S and T on X, respectively, then A, S and T satisfy the conditions (1.1), (1.2), and (2.1).

Further, the sequence $\{x_n\}$ of unique common fixed points of A_n, S_n and T_n converges to a unique common fixed point x of A, S and T if $\sup\{d(x_nx, z)\} < \infty$.

References

1. S. Gähler, *2-metrische Raume und ihr topologische Struktur*, Math. Nachr. **26** (1963), 115–148.
2. S. Gähler, *Über dei Uniformisierbarkeit 2-metrische Raume*, Math. Nachr. **28** (1965), 235–244.
3. S. Gähler, *Zur geometrie 2-metrische Raume*, Rev. Roumaine Math Pure Appl. **11** (1966), 665–667
4. K. Iséki, *A property of orbitally continuous mappings on 2-metric Spaces*, Math. Seminar Notes, Kobe Univ. **3** (1975), 131–132.
5. K. Iséki, *Fixed point theorems in 2-metric spaces*, Math. Seminar Notes, Kobe Univ. **3** (1975), 133–136.
6. K. Iséki, P. L. Sharma, and B. K. Sharma, *Contraction type mappings on 2-metric spaces*, Math. Japonica **21** (1976), 67–70.

7. M. D. Khan, *A study of fixed point theorems*, Doctoral Thesis, Aligarh Musmin University (1984).

8. M. S. Khan, *Convergence of sequences of fixed points in 2-metric spaces*, Indian J. Pure Appl. Math. **10** (1979), 1062–1067.

9. M. S. Khan, *On fixed point theorems in 2-metric spaces*, Publ. Inst. Math. (Beograd) (N.S) **41** (1980), 107–112.

10 M. S. Khan, and B. Fisher, *Some fixed point theorems for commuting mappings*, Marth. Nachr. **106** (1982), 323–326

11. M. S Khan and M. Swaleh, *Results concerning fixed points in 2-metric spaces*, Math Japonica **29** (1984), 519–525.

12. M. S. Khan, M. Imdad and M. Swaleh, *Asymptotically regular maps and sequences in 2-metric spaces*, Indian J. of Math **27** (1985), 81–88.

13. T. Kubiak, *Common fixed points of pairwise commuting mappings*, Math Nachr. **118** (1984), 123–127.

14. S L. Lal and A K. Singh, *An analogue of Banach's contraction principle for 2-metric Spaces*, Bull. Austral. Math Soc. **18** (1978), 137–143

15. S. L. Lal and A. K. Singh, *Invariant points of generalized nonexpansive mappings in 2-metric spaces*, Indian J. Math. **20** (1978), 71–76.

16. B A. Meade and S. P. Singh, *On common fixed point theorems*, Bull. Austral Math. Soc. **16** (1977), 49–53.

17. S. V R. Naidu and J R. Prasad, *Fixed point theorems in 2-metric spaces*, Indian J. Pure Appl. Math. **17** (1986), 974–993

18. H K. Pathak, *Weak* commuting mappings and fixed points*, Indian J. Pure Appl Math. **17** (1986), 201–211

19. B. Ram, *Existence of fixed points in 2-metric spaces*, Ph. D Thesis Garhwal Univ. (1982).

20. B E. Rhoades, *Contraction type mappings on a 2-metric space*, Math. Nachr **91** (1979), 151–154.

21. A K Sharma, *On fixed points in 2-metric spaces*, Math Seminar Notes, Kobe Univ. **6** (1978), 467–473.

22. A.K. Sharma, *A study of fixed points of mappings in metric and 2-metric space*, Ph. D. Thesis, Delhi Univ. (1979).

23. A. K. Sharma, *A generalization of Banach contraction principle in 2-metric spaces*, Math. Seminar Notes, Kobe Univ **7** (1979), 291–292

24. A K. Sharma, *A note on fixed points in 2-metric space*, Indian J Pure Appl Math. **11** (1980), 1580–1583.

25. S. L. Singh, *Some contraction type principles on 2-metric spaces and applications*, Math Seminar Notes, Kobe Univ. **7** (1979), 1–11.

26. S. L. Singh, *A fixed point theorem in a 2-metric spaces*, Math. Edu (Swain) **14** (1980), 53–54.

27 S. L. Singh B. M L Tiwari, and V K Gupta, *Common fixed points of commuting mappings in 2-metric spaces and application*, Math Nachr **95** (1980), 293–297.

28. S. L. Singh and B. Ram, *A note on the convergence of sequence of mappings and their common fixed points in a 2-metric space*, Math. Seminar Notes, Kobe Univ. **9** (1981), 181–185.
29. S. L. Singh and Virencha, *Coincidence theorems on 2-mtertic spaces*, Indian J. Phy. Nat. Sci. **2(13)** (1982), 32–35.

Department of Mathematics
Gyeongsang National University
Jinju 660–701, Korea

Department of Mathematics
Faculty of Science
Sultan Qaboos University
P.O.Box 32486, Al-Khod, Muscat
Sultanate of Oman