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APPLICATION FOR THE METHODS OF LINES TO NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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1. Introduction

Let $(H, \|\cdot\|, (\cdot,\cdot))$ denote a real Hilbert space and let I, S_t represent the intervals $[0,T], [-r,t]$, respectively where $0 < T < \infty$, $t \in I$ and $r \geq 0$. For a compact interval $J \subset R$, we shall denote by $C(J,H)$ the Banach space of all continuous functions from J into H endowed with the supremum norm $\|\cdot\|_{C(J,H)}$ and by $Lip(J,H)$ the class of all Lipschitz continuous functions from J into H. And we denote by $B_r(X)$ the closed ball $\{x \in X : \|x\|_X \leq r\}$ for positive constant r.

In this paper we apply the Method of Lines to establish the existence of unique strong solution of the following type of nonlinear abstract integro- differential equation :

$$(1.1) \qquad \begin{aligned} &\frac{du}{dt}(t) + Au(t) = G(t, u(t), F(u)(t)), \text{ for a.e. } t \in (0,T) \\ &u = \phi \text{ on } S_0, \ \phi \in Lip(S_0, H), \end{aligned}$$

where we assume the followings :

(H1) : The single valued nonlinear operator $A : D(A) \subset H \to H$ satisfies

(a) maximal monotonicity. i.e.,

$$(Au - Av, u - v) \geq 0 \quad \text{for all } u, v \in D(A)$$

and $R(I + A) = H$,

(b) $0 \in D(A)$

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(H2) : The mapping $G : I \times H \times H \to H$ satisfies

(a) $\|G(t,u,v)\| \leq M(1+\|u\|+\|v\|)$ for all $t \in I$ and $u, v \in H$ where M is a positive constant,

(b) For all $t_1, t_2 \in I$, $u_1, u_2, v_1, v_2 \in B_r(H)$

$$\|G(t_1,u_1,v_1) - G(t_2,u_2,v_2)\| \leq \|h(t_1) - h(t_2)\| + M_r(\|u_1 - u_2\| + \|v_1 - v_2\|)$$

where $h : I \to H$ is a continuous function of bounded variation and M_r is a positive constant depending on r.

(H3) : The nonlinear operator F is a Volterra operator (cf. [1]) which maps $C(S_T, H)$ into $C(S_T, H)$ satisfying

(a) $\|F(u)\|_{C(S_T,H)} \leq M(1+\|u\|_{C(S_T,H)})$ for all $u \in C(S_T, H)$

(b) $\|F(u) - F(v)\|_{C(S_T,H)} \leq M_r\|u - v\|_{C(S_T,H)}$ for all $u, v \in B_r(C(S_T, H))$ and

(c) there exists a continuous function $L : R_+ \to R_+$ such that

$$\|F(u)(t) - F(u)(s)\| \leq |t-s| L(\|u\|_{C(S_T,H)})(1 + \|\frac{du}{dt}\|_{L^\infty(S_t,H)})$$

for all $t, s \in I$ and $u \in Lip(S_T, H)$

We are now to show several previous results for the similar equations. They all have got the Method of Lines in common even having different conditions.

1. Necas [3] has solved the equation in Hilbert space

$$\begin{aligned} &\frac{du}{dt}(t) + Au(t) = f(t), \quad t \in (0,T) \\ &u(0) = u_0 \end{aligned} \tag{1.2}$$

where A is a maximal monotone operator, $u_0 \in D(A)$, and $f : [0,T] \to H$ is a continuous function of bounded variation.

2. Kartsatos and Zigler [2] have proved the existence of a unique weak solution of the following equation in a reflexive Banach space X whose dual is uniformly convex :

$$\begin{aligned} &\frac{du}{dt}(t) + Au(t) = G(t, u(t)), \quad t \in (0,T] \\ &u(0) = u_0 \end{aligned} \tag{1.3}$$

where $A : D(A) \subset X \to X$ is m-accretive operator and $G : [0,T] \times X \to X$ satisfies the Lipschitz-like condition

$$\|G(t,x) - G(s,y)\|_X \leq \|h(t) - h(s)\|_X + L\|x - y\|_X \tag{1.4}$$

for all $t, s \in [0,T]$ and all $x, y \in X$ with a continuous function of bounded variation h and a positive constant L. We note that the condition (4) is global in X.

3. Kacur [1] considered a paticular case $G(t, u(t), F(u)(t)) \equiv G(t, F(u)(t))$ of (1) in the Lion's set-up (i.e., there are reflexive space V and Hilbert space H such that $V \cap H$ is dense in V and H) with the following assumptions :

(A1) $A : V \to V^*$ is a maximal monotone operator satisfying

$$< Au, u > \geq \|u\| p(\|u\|) - C_1 \|u\|^2 - C_2 \tag{1.5}$$

Here $< \cdot, \cdot >$ denotes the duality product and $p : R_+ \to R_+$ satisfies the condition $p(s) \to \infty$ as $s \to \infty$.

(A2) $G : I \times H \to H$ satisfis the Lipschitz condition

$$\|G(t,u) - G(s,v)\| \leq C[|t-s|(1 + \|u\|) + \|u - v\|] \tag{1.6}$$

for all $t, s \in I$ and $u, v \in H$.

(A3) The Volterrl operator F satisfies

$$\|F(u) - F(v)\|_{C(S_T,H)} \leq \|u - v\|_{C(S_T,H)}$$

for all $u, v \in C(S_T, H)$, and

$$\|F(u)(t) - F(u)(s)\| \leq |t-s| L(\|u\|_{C(S_T,H)})(1 + \|\frac{du}{dt}\|_{L^\infty(S_T,H)})$$

for all $t, s \in I, s < t$, and $u \in Lip(S_T, H)$

In this paper, we weaken the global Lipschitz conditions-(1.4), (A2) and (A3) and take more general nonlinear mapping G into consideration. Moreover, we do not assume any coercivity on the operator A as in (A1). Instead, we assume $0 \in D(A)$ which is not a very strong condition. It is obvious that (A2) and (A3) imply our hypotheses (H2) and (H3). But the reverse is not true in general.

2. Main Results

To apply the Method of Lines, we follow the following procedure : For any positive integer n we consider a partition $\{t_j^n\}$ defined by $t_j^n = j \cdot h$, $h = \dfrac{T}{n}$. Setting $u_0^n = \phi(0)$, we successively solve for $u \in D(A)$ the equation

$$\text{(2.1)} \qquad \frac{u - u_{j-1}^n}{h} + Au = G(t_j^n, u_{j-1}^n, F(\tilde{u}_{j-1}^n)(t_j^n))$$

where

(2.2)

$$\tilde{u}_{j-1}^n = \begin{cases} \phi \text{ on } S_0, \\ \phi(0) \text{ on } [0,h], \\ u_{i-1}^n + \frac{1}{h}(t - t_{i-1}^n)(u_i^n - u_{i-1}^n) \text{ for } t \in [t_{i-1}^n, t_i^n],\ i = 1, \ldots, j, \\ u_{j-1}^n \text{ on } [t_j^n, T]. \end{cases}$$

The existence of unique $u_j^n \in D(A)$ satisfying (2.1) is a consequence of maximal monotonicity of the operator A. We first show, using (H1), (H2)-(a), and (H3)-(a), that $\|u_j^n\| \le M$ for all n and $j = 1, 2, \ldots, n$ where M is a positive constant independent of j, h, and n. Then we prove that $\dfrac{1}{h}\|u_j^n - u_{j-1}^n\| \le M$. After all we define a sequence $\{z^n\} \subset Lip(S_T, H)$ given by

$$\text{(2.3)} \qquad z^n(t) = \begin{cases} \phi(t) \text{ for } t \in S_0, \\ u_{j-1}^n + \frac{1}{h}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n) \text{ for } t \in (t_{j-1}^n, t_j^n] \end{cases}$$

and a sequence $\{u^n\}$ of step functions mapping from $(-h, T]$ into H given by

$$\text{(2.4)} \qquad u^n(t)) = \begin{cases} \phi(0) & \text{for } t \in (-h, 0], \\ u_j^n & \text{for } t \in (t_{j-1}^n, t_j^n]. \end{cases}$$

After proving some a priori estimate for $\{z^n\}$ and $\{u^n\}$ we establish the following main result.

THEOREM 1. *Let hypotheses (H1)-(H3) be satisfied and let $\phi(0) \in D(A)$. Then there exists unique strong solution $u \in Lip(S_T, H)$ of (1.1) in the sense that $u = \phi$ on S_0, $\frac{du}{dt} \in L^\infty(I,H)$, $Au \in L^\infty(I,H)$ and equation (1.1) is satisfied a.e. on I.*

3. Proofs of Main Result

We shall denote by

$$z_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \qquad g_j^n = G(t_j^n, u_{j-1}^n, F(\tilde{u}_{j-1}^n)(t_j^n))$$

for $j = 1, 2, \ldots, n$. For notational convience, we shall supress the superscript n sometimes. In the sequel, we denote by M a generic constant independent of j, h, and n.

LEMMA 1. *Let the hypotheses (H1), (H2)-(a), and (H3)-(a) be satisfied and let $\phi(0) \in D(A)$. Then $\|u_j\| \leq M$ for all n and $j = 1, 2, \ldots, n$.*

Proof. From (H1), $(Au - A0, u) \geq 0$ which implies that $(Au, u) \geq -M_1\|u\|^2 - M_2$, where M_1 and M_2 are positive constants. Now, from (2.1) we have

$$(\frac{u_j - u_{j-1}}{h}, v) + (Au_j, v) = (g_j, v) \tag{3.1}$$

for all $v \in H$. We put $v = hu_j$ to obtain

$$\frac{1}{2}\|u\|^2 - \frac{1}{2}\|u_{j-1}\|^2 - M_1 h\|u_j\|^2 - M_2 h \leq h\|g_j\|\,\|u_j\|. \tag{3.2}$$

Using (H2)-(a) and (H3)-(a), we get

$$\|u_i\|^2 - \|u_{i-1}\|^2 \leq Mh(1 + \max_{1\leq k\leq i} \|u_k\|^2)$$

for $i = 1, 2, \ldots, n$. Summing up the above inequality for $i = 1$ to j, we obtain

$$\|u_j\|^2 \leq M(1 + \sum_{i=1}^{j} \max_{1\leq k\leq i} \|u_k\|^2).$$

Hence we have

$$\max_{1\leq k\leq j} \|u_k\|^2 \leq M(1 + h\sum_{i=1}^{j} \max_{1\leq k\leq i} \|u_k\|^2).$$

Application of Grownwall's Lemma gives the required result.

LEMMA 2. *In addition to the hypotheses of Lemma 1, we assume that (H2)-(b) and (H3)-(b)(c) are also satisfied. Then* $\|z_j\| \leq M$ *for all* n *and* $j = 1, 2, \ldots, n$.

Proof. From (3.1) for $j = 1$ and $v = z_1$, we have $\|z_1\| \leq M$ for all n. Also, we get

$$(z_j, v) + (Au_j - Au_{j-1}, v) = (z_{j-1}, v) + (g_j - g_{j-1}, v)$$

for all $v \in H$ and $j = 2, 3, \ldots, n$. Putting $v = z_j$, we have $\|z_j\| \leq \|z_{j-1}\| + \|g_j - g_{j-1}\|$. Using Lemma 1, (H2)-(b), and (H3)-(b)(c), we obtain an estimate

$$\begin{aligned}\|g_i - g_{i-1}\| \leq \|h(t_i) - h(t_{i-1})\| + Mh(\|z_{i-1}\| \\ + \|\frac{\tilde{u}_{i-1} - \tilde{u}_{i-2}}{h}\|_{C(S_T, H)} + 1 + \|\frac{d\tilde{u}_{i-2}}{dt}\|_{L^\infty(S_t, H)}) \\ + \|h(t_i) - h(t_{i-1})\| + Mh(1 + \max_{1 \leq k \leq i} \|z_k\|).\end{aligned}$$

Therefore we have for $i = 1, 2, \ldots, n$;

$$\|z_i\| - \|z_{i-1}\| \leq \|h(t_i) - h(t_{i-1})\| + Mh(1 + \max_{1 \leq k \leq i} \|z_k\|).$$

Summing up the inequality for $i = 1$ to j, we obtain

$$\|z_j\| \leq M(1 + h \sum_{i=1}^{j} \max_{1 \leq k \leq i} \|z_k\|).$$

Proceeding similarly as in Lemma 1, we get the required result.

REMARK 1. Lemma 1 and Lemma 2 imply that

$$\|z^n(t) - u^n(t)\| \leq \frac{M}{n}, \qquad \|z^n(t) - z^n(s)\| \leq M|t - s|$$

and $\|z^n(t)\| + \|u^n(t)\| \leq M$ for all n and $t, s \in I$.

Again, for the notational convenience, we shall denote by

(3.3) $w^n(t) = G(t_j^n, u_{j-1}^n, F(\tilde{u}_{j-1})(t_j^n))$, for $t \in (t_{j-1}^n, t_j^n]$, $1 \leq j \leq n$.

Then (2.1) can be rewritten in the form

$$\frac{d^-}{dt}z^n(t) + Au^n(t) = w^n(t), \text{ for } t \in (0,T], \tag{3.4}$$

where $\frac{d^-}{dt}$ denotes the left-derivative. Also, we have

$$\int_0^t Au^n(s)ds = u_0 - z^n(t) + \int_0^t w^n(s)ds. \tag{3.5}$$

LEMMA 3. *There exists $u \in Lip(S_T, H)$ such that $u = \phi$ on S_0 and $z^n \to u$ in $C(S_T, H)$.*

Proof. From (3.4) for $t \in (0,T]$ and for all $v \in H$ we have

$$(\frac{d^-}{dt}z^n(t) - \frac{d^-}{dt}z^m(t), v) + (Au^n(t) - Au^m(t), v) = (w^n(t) - w^m(t), v).$$

For $v = u^n(t) - Au^m(t)$, using monotonicity of A and the fact

$$2(\frac{d^-}{dt}z^n(t) - \frac{d^-}{dt}z^m(t), z^n(t) - z^m(t)) = \frac{d^-}{dt}\|z^n(t) - z^m(t)\|^2,$$

we get

$$\begin{aligned}
&\frac{1}{2}\frac{d^-}{dt}\|z^n(t) - z^m(t)\|^2 \\
&\leq (\frac{d^-}{dt}z^n(t) - \frac{d^-}{dt}z^m(t) - w^n(t) - w^m(t), z^n(t) - u^n(t) - z^n(t) + u^m(t)) \\
&\qquad + (w^n(t) - w^m(t), u^n(t) - u^m(t)).
\end{aligned}$$

Now,

$$\|w^n(t) - w^m(t)\| \leq \epsilon_{nm}(t) + \|z^n - z^m\|_{C(S_T,H)},$$

where

$$\begin{aligned}
\epsilon_{nm}(t) = \|h^n(t) - h^m(t)\| + M(|\psi^n(t) - \psi^m(t)| \\
+ \|u^n(t) - z^n(t - \frac{T}{n})\| + \|u^m(t) - z^m(t - \frac{T}{m})\| \\
+ \|\tilde{u}^n_{n-1} - z^n\|_{C(S_t,H)} + \|\tilde{u}^m_{m-1} - z^m\|_{C(S_t,H)}),
\end{aligned}$$

for $h^n(t) = h(t_j^n)$, $\psi^n(t) = t_j^n$, $t \in (t_{j-1}^n, t_j^n]$, $h^n(0) = h(0)$, $\psi^n(0) = 0$. Clearly $h^n(t) \to h(t)$ and $\psi^n(t) \to t$ uniformly on I as $n \to \infty$. Hence we have the estimate

$$\frac{d^-}{dt}\|z^n(t) - z^m(t)\|^2 \leq M(\epsilon_{nm} + \|z^n - z^m\|^2_{C(S_t,H)})$$

where $\{\epsilon_{nm}\}$ is a sequence of numbers such that $\epsilon_{nm} \to 0$ on I as $n, m \to \infty$. Integrating over $(0, s)$ and taking supremum for $s \in (0, t)$ on both sides, we get

$$\|z^n - z^m\|^2_{C(S_t,H)} \leq M(\epsilon_{nm} \cdot T + \int_0^t \|z^n - z^m\|^2_{C(S_s,H)} ds).$$

Applying Grownwall's Lemma, we conclude that there exists $u \in C(S_T, H)$ such that $z^n \to u$ in $C(S_T, H)$. Obviously, $u = \phi$ on S_0 and from Remark 1, $u \in Lip(S_T, H)$.

Proof of Theorem 1. Proceding similarly as in [2], it is easy to show $u(t) \in D(A)$ for $t \in I$, $Au^n(t) \rightharpoonup Au(t)$ (weakly), and $Au(t)$ is weakly continuous in t. From (3.5), for every $v \in H$, we have

$$\int_0^t (Au^n(s), v)ds = (u_0, v) - (z^n(t), v) + \int_0^t (w^n(s), v)ds.$$

Using Lemma 3 and bounded convergence theorem, we pass through the limit for $n \to \infty$ to obtain

(3.6)

$$\int_0^t (Au(s), v)ds = (u_0, v) - (u(t), v) + \int_0^t (G(s, u(s), F(u)(s), v)ds.$$

Since $Au(t)$ is Bochner integrable, (3.6) implies that

$$\frac{du}{dt}(t) + Au(t) = G(t, u(t), F(u)(t)) \quad \text{for a.e. } t \in I.$$

Now, we show the uniqueness of strong solution. Let u_1 and u_2 be two strong solutions of equation (1.1). Let $u = u_1 - u_2$ and let $r = \max_{t\in[0,T]}\{\|u_1\|, \|u_2\|\}$. Then for a.e. $t \in I$, we have

$$(\frac{du}{dt}(t), u(t)) + (Au_1(t) - Au_2(t), u(t))$$
$$= (G(t, u_1(t), F(u_1)(t)) - G(t, u_2(t), F(u_2)(t)), u(t)).$$

Hypotheses (H1), (H2), and (H3) imply that

$$\frac{d}{dt}\|u(t)\|^2 \leq M_r \|u\|^2_{C(S_t,H)} \quad \text{for a.e. } t \in I,$$

where M_r is a positive constant depending on r. Integrating over $(0, s)$ and taking supremum both sides for $s \in (0, t)$ we get

$$\|u\|^2_{C(S_t,H)} \leq M_r \int_0^t \|u\|^2_{C(S_s,H)} ds.$$

From Grownwall's Lemma, $u(t) \equiv 0$ on I.

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