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REGULAR ACTION IN A UNIT–REGULAR RING

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1. Introduction and Basic Definitions

Let R be a ring with identity and let G denote the group of units of R and X denote the set of nonzero, nonunits of R. We call the action, $(g, r) \to gr$ from $G \times R$ to R, the regular action. Clearly, X is invariant under the action.

If $f : G \times X \to X$ is a group action on X, for each x in X, we define the orbit $0(x)$ by, $0(x) = \{f(g, x) | g \in G\}$. G is said to be transive on X if there is an x in X with $0(x) = X$ and G is said to be half-transive on X if G is transive on X or $0(x)$ is finite and $|0(x)| = |0(y)| > 1$ for all x and y in X.

It is easyly shown that if a ring R with identity has finite nonzero number of nontrivial idempotents of R, then the number is even. An idempotent e in a ring R is called primitive if it cannot be written as the sum of two orthogonal nonzero idempotents, and is called central if it is contained in the center of R.

A ring R is called regular (resp. unit-regular) if for each x in R, there is an elememt u in R (resp. unit u in G) such that $xux = x$. A regular ring R is abelian if all idempotents in R are central. It was already shown in [2,Corollary 4.2, p.38] that every abelian regular ring is unit-regular.

In section 2, we show that if R is a unit-regular ring and R has no nontrivial idempotents, then R is a division ring. We also show that in case that R is a unit-regular ring such that G acts on X by the regular action and R has a finite nonzero number of nontrivial idempotents, if $|0(x)| = 1$ for all x in X or G is half-transitive on X, then R is finite. In particular, if a unit-regular ring R has two or four nontrivial idempotents which are central, then R is isomorphic to the product of two finite fields which are isomorphic to each other.

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2. The regular action in a unit-regular ring

The following theorem has been proved in [4].

THEOREM 2.1. *If R is any ring having only $n+1$ left (right) zero-divisors, where n is positive integer, then R is necessarily finite and does not contain more than $(n+1)^2$ elements.*

LEMMA 2.2. *Let R be a ring with unity 1 and let E be the set of all nontrivial idempotents in R. If E is a nonempty finite set, then $|E|$ is even.*

Proof. It is clear that for each $e \in E$, $1-e \in E$. Assume that $e = 1-e$. Then $1 = 2e$. If $\mathrm{char}(R) = 2$, then $1 = 2e = 0$, a contradiction. If $\mathrm{char}(R) \neq 2$, then e is invertible, a contradiction. Hence for each $e \in E$, $e \neq 1-e$. Therefore $|E|$ is even.

LEMMA 2.3. *Let R be a unit-regular ring. If R has no nontrivial idempotents, then R is a division ring.*

Proof. For any nonzero X in R, there is a unit u in R such that $xux = x$. Then xu and ux are idempotents of R. By assumption, $xu = ux = 1$. Therefore R is a division ring.

LEMMA 2.4. *Let R be a unit-regular ring. Then for all $a \in R$, a is a unit or a is a zero divisor.*

Proof. Suppose $a \in R$ is not unit. Since R is unit-regular, there exists a unit $u \in R$ such that $aua = a$, so $0 = a(ua-1) = (au-1)a$. If $ua = 1$, then $a = u^{-1}$, a contradiction. Hence $ua - 1 \neq 0$, which means that a is right zero-divisor. Similarly, we can show that a is left zero-divisor.

LEMMA 2.5. *Let R be a unit-regular ring such that G acts on X where G is the set of units in R and X is the set of nonzero, nonunits in R. If $|0(x)| = 1$ for all $x \in X$, then every $x \in X$ is nontrivial idempotent of R.*

Proof. Assume that there exists a $x \in X$, which is not nontrivial idempotent. Then there exists a unit $u \in G$ such that ux is a nontrivial idempotent of R. Since $|0(x)| = 1$ for all $x \in X$, $ux = x$, a contradiction.

Corollary 2.6. *Let R be a unit-regular ring such that G acts on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite and $|0(x)| = 1$ for all $x \in X$, then R is a finie ring.*

Proof. It follow from Theorem 2.1 and Lemma 2.5.

The following Lemma has been proved in [1].

Lemma 2.7. *Let R be a ring with identity such that G is half-transitive on X by the regular action. If R is not a local ring and A/J (J is the Jacobson radical of R) contains a nontrivial idempotent, then G is a finite group.*

Lemma 2.8. *Let R be a unit-regular ring such that G acts on X by the regular action and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then there exist $x_1, x_2, \cdots, x_n \in X$ such that $X = 0(x_1) \cup 0(x_2) \cdots \cup 0(x_n)$. In particular, if $|0(x)| = |0(y)| = s$ for all $x, y \in X$ and for some positive integer s, then $|X| = s|E|$.*

Proof. For all $x \in X$, there exixts $u_x \in G$ such that $u_x x$ is a nontrivial idempotent in R. Since $|E|$ is finite, say $u_1x_1, u_2x_2, \cdots, u_nx_n$ where $u_i \in G$ and $x_i \in X (1 \leq i \leq n = |E|)$, for each $x \in X \setminus \{x_1, x_2, \cdots, x_n\}$, $u_x x = u_i x_i$ for some i. So $x = u_x^{-1} u_i x_i$ and then $x \in 0(x_i)$. Hence $X = 0(x_1) \cup 0(x2) \cup \cdots \cup 0(x_n)$ and so $|X| = |0(x_1)| + |0(x_2)| + \cdots + |0(xn)|$. In particular, if $|0(x)| = |0(y)| = s$ for all $x, y \in R$, then $|X| = sn = s|E|$.

Theorem 2.9. *Let R be a unit-regular ring such that G is half-transitive on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then R is a finite ring.*

Proof. It follows from Lemma 2.2 and Lemma 2.7 that G is finite. Since G is finite, if G is transitive on X, then X is finite. So by Theorem 2.1, R is finite ring. Suppose that G is not transitive on X but G is half-transitive on X. By lemma 2.3, $X = 0(x1) \cup 0(x2) \cup \cdots \cup 0(xn)$ for some $x_i \in X (1 \leq i \leq n = |E|)$. Since $0(x_i)$ is finite, $|X| = |0(x_1)| + |0(x_2)| + \cdots + |0(x_n)|$, so X is finite. Therefore, by Theorem 2.1, R is a finite ring.

Lemma 2.10. *Let R be a ring with identity. Then the idempotent $e \neq 0$ of R is primitive if and only if eRe contains no idempotents other than 0 and e.*

Proof. ($\Rightarrow$) Suppose that $e \neq 0$ of R is primitive. Let $f \in eRe$ be an idempotent. Then $f = ere$ for some $r \in R$. Clearly $ef = f = fe$ and $(e-f)^2 = e - f$ and also $(e-f)f = ef - f2 = f - f = 0$ and $f(e-f) = 0$. So $e = (e-f) + f$, and $(e-f)f = f(e-f) = 0$. Since e is primitive, $e - f = 0$ or $f = 0$. Hence eRe contains no idempotents other than 0 and e.

($\Leftarrow$) Suppose that eRe contains no idempotents other than 0 and e. Assume that $e = e_1 + e_2$ where $e_1e_2 = e_2e_1 = 0$, $e_1^2 = e_1$ and $e_2^2 = e_2$. Since $e_1e = e_1 \in Re$ and $ee_1 = e_1 \in eR$,

$$e_1 = e_1^2 = e_1e_1 = (ee_1)(e_1e) \in (eR)(Re) \subset eRe.$$

Similarly, $e_2 \in eRe$. By assumption, since eRe contains no idempotent other than 0 and e, $e_1 = 0$ or $e_1 = e$. Hence e is primitive.

LEMMA 2.11. *Let R be a regular ring with identity. Then e is a primitive idempotent in R iff eRe is a division ring.*

Proof. ($\Rightarrow$) If e is a primitive idempotent in R, by Lemma 2.10, eRe contains no idempotent other than 0 and e. For $x \neq 0 \in eRe$, consider right ideal xR of R. Since R is regular, $xR = (t)$ for some idempotent $t \in eR$. By assumption, $t = e$ and so $xr = e$ for some $r \in R$. We can note that $(rx)(rx) = r(xr)x = rex = rx$ and also $rx = e$. Hence x is invertible in eRe, and so eRe is a division ring.

($\Leftarrow$) It is enough to show that eRe has no idempotents other than e and 0 by Lemma 2.10. Let $f = f^2 \in eRe$. Then $f(e-f) = 0$. Since eRe is a division ring, $f = 0$ or $e - f = 0$.

LEMMA 2.12. *Let R be a regular ring with identity let E be the set of nontrivial idempotents in R. If $|E| = 2$ or 4, then every element of E is primitive.*

Proof. If $E = 2$, then clearly every element of E is primitive. Suppose that $E = 4$. Let $e, 1-e, f$, and $1-f$ be all dictinct elements of E. We will show that e is primitive idempotent of R. Assume that e is not primitive idempotent of R. Then we have two possibilities that e can be written as the sum of two orghogonal nonzero idempotents say, $e = (1-e) + f$ or $e = (1-e) + (1-f)$. Then the equality $e = (1-e) + f$ implies that $e = f$, a contradiction. Also the equality

$e = (1 - e) + (1 - f)$ implies that $1 - e = f$, a contradiction. Similarly we can show that $1 - e, f, 1 - f$ are primitive.

Recall that a regular ring R is abelian if every idempotent is central.

THEOREM 2.13. *Let R be an abelian regular ring such that G is half-transitive on X and let E be the set of nontrivial idempotents in R. If $E = 2$ or 4, then R is a direct sum of two finite fields F_1 and F_2 where F_1 is isomorphic to F_2.*

Proof. By Lemma 2.12 and assumption, all element of E are primitive and central. Thus $R = eR \oplus (1 - e)R$ for some $e \in E$. By Lemma 2.9, R is finite. By Lemma 2.11, since e is primitive, eR and $(1 - e)R$ are finite fields. Clearly, the function $er \to (1 - e)r$ defined by $er \to (1 - e)r$ for all $er \in eR$, is a ring isomorphism. Hence we have the result.

COROLLARY 2.14. *Let R be a unit-regular ring such that G is transitive on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then $R \approx M_1(F_1) \oplus M_2(F_2) \oplus \cdots \oplus M_k(F_k)$ with $|F_1|^{n_1} \cdot \cdots \cdot |F_k|^{n_k} \leq |G| + 1$ where $M_i(F_i)$ is the ring of all $n_i \times n_i$ matrices over a finite field F_i for $i = 1, 2, \cdots, k$ and k is some positive integer.*

Proof. By Theorem 2.9, R is finite. Since G is transitive on X, $|X| \leq |G|$. Since R is finite and semisimple, by the Wedderburn - Artin Theorem, we have that $R \approx M_1(F_1) \oplus M_2(F_2) \oplus \cdots \oplus M_k(F_k)$ where $M_i(F_i)$ is the ring of all $n_i \times n_i$ matrices over a finite field F_i for $i = 1, 2, \cdots, k$ and k is some positive integer. Hence it follows from Theorem 2.1 that $|F1|^{n_1} \cdot \cdots \cdot |F_k|^{n_k} \leq |G| + 1$.

COROLLARY 2.15. *Let R be a abelian regular ring such that G is transitive on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then $R \approx F_1 \oplus F_2 \oplus \cdots \oplus F_k$ with $|R| \leq (|G| + 1)^2$ where F_i is a finite field for $i = 1, 2, \cdots, k$ and k is some positive integer.*

Proof. Since abelian regualr ring R is unit-regualr and every idempotents in R is central, it must be $n_i = 1$ in the proof of Corollary 2.14 for each $i = 1, 2, \cdots, k$. Hence we have the result.

Corollary 2.16. *Let R be a unit-regular ring such that G is not transitive but is half-transitive on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then $R \approx M_1(F_1) \oplus M_2(F_2) \oplus \cdots \oplus M_k(F_k)$ with $|F_1|^{n_1} \cdot \cdots \cdot |F_k|^{n_k} \leq (s|E|+1)$ where $|0(x)| = s$ for all $x \in X$ and $M_i(F_i)$ is the ring of all $n_i \times n_i$ matrices over a finite field F_i for $i = 1, 2, \cdots, k$ and k is some positive integer.*

Proof. By Theorem 2.9, R is also finite. Since G is not transitive on X but is half-transitive on X, by Theorem 2.8, $|X| = s|E|$ where $|0(x)| = s$ for all $x \in X$. Since R is finite and semisimple, as proof in the Corollary 2.14, $R \approx M_1(F_1) \oplus M_2(F_2) \oplus \cdots \oplus M_k(F_k)$ where $M_i(F_i)$ is the ring of all $n_i \times n_i$ matrices over a finite field F_i for $i = 1, 2, \cdots, k$ and k is some positive integer. Hence it follows from Theorem 2.1 that $|F_1|n_1 \cdot \cdots \cdot |F_k|^{n_k} \leq (s|E|+1)$.

Corollary 2.17. *Let R be an abelian regular ring such that G is not transitive on X but half-transitive on X and let E be the set of nontrivial idempotents in R. If $0 \neq |E|$ is finite, then $R \approx F_1 \oplus F_2 \oplus \cdots \oplus F_k$ with $|R| \leq (s|E|+1)^2$ where F_i is a finite field for $i = 1, 2, \cdots, k$ and k is some positive integer and $s = |0(x)|$ for all $x \in X$.*

Proof. Similar to the proof of Corollary 2.15.

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