Pusan Kyŏngnam Math. J. 7(1991), No. 2, pp. 143–146

A NOTE ON LEFT REGULAR MAPS IN BCK–ALGEBRAS

Y. B. Jun and S. M. Hong

An algebraic system $< X, *, 0 >$ of type (2, 0) is called a *BCK-algebra* if it satisfies the following conditions:

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $0 * x = 0$,
(5) $x * y = 0 = y * x$ implies that $x = y$,

for every $x, y, z \in X$.

If we define $x \leq y$ as $x * y = 0$, then $< X, \leq >$ is a partially ordered set. A BCK-algebra X is said to be *positive implicative* (resp. *implicative*) if $(x * z) * (y * z) = (x * y) * z$ (resp. $x * (y * x) = x$) for all $x, y \in X$. Every implicative BCK-algebra is positive implicative.

Definition 1. ([1]). Let X be a BCK-algebra. Then a self map $\alpha : X \to X$ is said to be *left regular* if $\alpha(x * y) = \alpha(x) * y$ for all $x, y \in X$.

By definition, we have $\alpha(0) = 0$ and $\alpha(x) \leq x$ for every $x \in X$.

Example 2. ([2]). Let X be a partially ordered set with the least element 0 such that every pair of non-zero distinct elements is incomparable. We define the operation $*$ on X as follows:

$$x * y = \begin{cases} 0, & \text{if } x \leq y \\ x, & \text{otherwise.} \end{cases}$$

Under this operation $*$, X is an implicative BCK-algebra.

Received November 11, 1991

PROPOSITION 3. *Let X be the BCK-algebra of Example 2. If we define a map $\alpha : X \to X$ by $\alpha(a) = a$ and $\alpha(x) = 0$ for a fixed non-zero element a and all $x \neq a \in X$, then α is a left regular map of X.*

Proof. It can be easily checked that $\alpha(0*x) = \alpha(0)*x$ and $\alpha(x*0) = \alpha(x)*0$ for every $x \in X$. Let x be any non-zero element of X. If $x = a$, then $\alpha(x * a) = \alpha(a * a) = \alpha(0) = 0 = a * a = \alpha(a) * a = \alpha(x) * a$ and $\alpha(a * x) = \alpha(a * a) = \alpha(0) = 0 = a * a = \alpha(a) * a = \alpha(a) * x$. If $x \neq a$, then $\alpha(x * a) = \alpha(x) = 0 = 0 * a = \alpha(x) * a$ and $\alpha(a * x) = \alpha(a) = a = a * x = \alpha(a) * x$. Let x and y be any non-zero elements of X such that $x \neq a$ and $y \neq a$. Then $\alpha(x * y) = \alpha(x) = 0 = 0 * y = \alpha(x) * y$. This completes the proof.

The following result is easily seen.

PROPOSITION 4. *If $\alpha, \beta : X \to X$ are left regular maps of a BCK-algebra X, then so is $\alpha\beta$.*

PROPOSITION 5. *Let X be a positive implicative BCK-algebra and let α be a left regular map of X. If we define a map $\alpha' : X \to X$ by $\alpha'(x) = x * \alpha(x)$ for all $x \in X$, then α' is left regular.*

Proof. Let x and y be any elements of X. Then

$$\begin{aligned} \alpha'(x * y) &= (x * y) * \alpha(x * y) \\ &= (x * y) * (\alpha(x) * y) \\ &= (x * \alpha(x)) * y \\ &= \alpha'(x) * y, \end{aligned}$$

which completes the proof.

We denote the set of all left regular maps of a BCK-algebra X by $LR(X)$. Following Proposition 4, $LR(X)$ is closed under composition. Moreover the associative law holds, and clearly the identity map $1 : X \to X$ is a left regular map. Thus we have

THEOREM 6. *Let X be a BCK-algebra. Then $(LR(X), \circ)$ is a monoid.*

We refer the reader to [3] and [4] for details on ideals and quotient algebras in BCK-algebras.

PROPOSITION 7. *Let I be an ideal of a BCK-algebra X and let $\alpha : X \to X$ be a left regular map. Then the map $\overline{\alpha} : X/I \to X/I$ defined by $\overline{\alpha}(C_x) = C_{\alpha(x)}$ for all $C_x \in X/I$ is left regular.*

Proof. We have that

$$\begin{aligned}\overline{\alpha}(C_x * C_y) &= \overline{\alpha}(C_{x*y}) \\ &= C_{\alpha(x*y)} \\ &= C_{\alpha(x)*y} \\ &= C_{\alpha(x)} * C_y \\ &= \overline{\alpha}(C_x) * C_y\end{aligned}$$

for every $C_x, C_y \in X/I$. This completes the proof.

THEOREM 8. *Let X be a positive implicative BCK-algebra. Then every left regular map α of X has a fixed point, i.e., $\alpha(y) = y$ for some $y \in X$.*

Proof. Note that X satisfies the identities

(6) $(x * (y * x)) * (x * y) = (y * (y * x)) * (x * y)$,
(7) $x * 0 = x$.

Hence we have

$$(\alpha(x) * (x * \alpha(x))) * (\alpha(x) * x) = (x * (x * \alpha(x))) * (\alpha(x) * x)$$

for each $x \in X$. It follows from $\alpha(x) \leq x$ and (7) that

$$\alpha(x * (x * \alpha(x))) = \alpha(x) * (x * \alpha(x)) = x * (x * \alpha(x))$$

for all $x \in X$, which completes the proof.

References

1. W.H. Cornish, *A multiplier to approach to implicative BCK-algebras*, Math. Seminar Notes **8** (1980), 157–169
2. K. Iséki, *Remarks on contrapositionally complemented BCK-algebras (NB BCK-algebras)*, Math Seminar Notes **7** (1979), 633–638.
3. K. Iséki and S Tanaka, *Ideal theory of BCK-algebras*, Math. Japonica **21** (1976), 351–366.

4. K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica **23** (1978), 1–26.

Department of Mathematics
Gyengsang National University
Chinju 660–701, Korea