A NOTE ON LEFT REGULAR MAPS IN BCK-ALGEBRAS

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An algebraic system \( \langle X, *, 0 \rangle \) of type (2, 0) is called a BCK-algebra if it satisfies the following conditions:

1. \( ((x * y) * (x * z)) * (z * y) = 0 \),
2. \( (x * (x * y)) * y = 0 \),
3. \( x * x = 0 \),
4. \( 0 * x = 0 \),
5. \( x * y = 0 \) implies that \( x = y \),

for every \( x, y, z \in X \).

If we define \( x \leq y \) as \( x * y = 0 \), then \( \langle X, \leq \rangle \) is a partially ordered set. A BCK-algebra \( X \) is said to be positive implicative (resp. implicative) if \( (x * z) * (y * z) = (x * y) * z \) (resp. \( x * (y * x) = x \)) for all \( x, y \in X \). Every implicative BCK-algebra is positive implicative.

DEFINITION 1. ([1]). Let \( X \) be a BCK-algebra. Then a self map \( \alpha : X \rightarrow X \) is said to be left regular if \( \alpha(x * y) = \alpha(x) * y \) for all \( x, y \in X \).

By definition, we have \( \alpha(0) = 0 \) and \( \alpha(x) \leq x \) for every \( x \in X \).

EXAMPLE 2. ([2]). Let \( X \) be a partially ordered set with the least element 0 such that every pair of non-zero distinct elements is incomparable. We define the operation \( * \) on \( X \) as follows:

\[
x * y = \begin{cases} 
0, & \text{if } x \leq y \\
x, & \text{otherwise.}
\end{cases}
\]

Under this operation \( * \), \( X \) is an implicative BCK-algebra.
PROPOSITION 3. Let $X$ be the BCK-algebra of Example 2. If we define a map $\alpha : X \to X$ by $\alpha(a) = a$ and $\alpha(x) = 0$ for a fixed non-zero element $a$ and all $x \neq a \in X$, then $\alpha$ is a left regular map of $X$.

Proof. It can be easily checked that $\alpha(0*x) = \alpha(0)*x$ and $\alpha(x*0) = \alpha(x)*0$ for every $x \in X$. Let $x$ be any non-zero element of $X$. If $x = a$, then $\alpha(x*a) = \alpha(a*a) = \alpha(0) = 0 = a*a = \alpha(a)*a = \alpha(x)*a$ and $\alpha(a*x) = \alpha(a*a) = \alpha(0) = 0 = a*a = \alpha(a)*a = \alpha(a)*x$. If $x \neq a$, then $\alpha(x*a) = \alpha(x) = 0 = 0*a = \alpha(x)*a$ and $\alpha(a*x) = \alpha(a) = a = a*a = \alpha(a)*x$. Let $x$ and $y$ be any non-zero elements of $X$ such that $x \neq a$ and $y \neq a$. Then $\alpha(x*y) = \alpha(x) = 0 = 0*y = \alpha(x)*y$. This completes the proof.

The following result is easily seen.

PROPOSITION 4. If $\alpha, \beta : X \to X$ are left regular maps of a BCK-algebra $X$, then so is $\alpha \beta$.

PROPOSITION 5. Let $X$ be a positive implicative BCK-algebra and let $\alpha$ be a left regular map of $X$. If we define a map $\alpha' : X \to X$ by $\alpha'(x) = x*\alpha(x)$ for all $x \in X$, then $\alpha'$ is left regular.

Proof. Let $x$ and $y$ be any elements of $X$. Then

$$\alpha'(x*y) = (x*y)*\alpha(x*y)$$
$$= (x*y)*(\alpha(x)*y)$$
$$= (x*\alpha(x))*y$$
$$= \alpha'(x)*y,$$

which completes the proof.

We denote the set of all left regular maps of a BCK-algebra $X$ by $LR(X)$. Following Proposition 4, $LR(X)$ is closed under composition. Moreover the associative law holds, and clearly the identity map $1 : X \to X$ is a left regular map. Thus we have

THEOREM 6. Let $X$ be a BCK-algebra. Then $(LR(X), \circ)$ is a monoid.

We refer the reader to [3] and [4] for details on ideals and quotient algebras in BCK-algebras.
PROPOSITION 7. Let $I$ be an ideal of a BCK-algebra $X$ and let $\alpha : X \rightarrow X$ be a left regular map. Then the map $\bar{\alpha} : X/I \rightarrow X/I$ defined by $\bar{\alpha}(C_x) = C_{\alpha(x)}$ for all $C_x \in X/I$ is left regular.

Proof. We have that

$$\bar{\alpha}(C_x \ast C_y) = \bar{\alpha}(C_{x\ast y})$$
$$= C_{\alpha(x\ast y)}$$
$$= C_{\alpha(x)} \ast y$$
$$= C_{\alpha(x)} \ast C_y$$
$$= \bar{\alpha}(C_x) \ast C_y$$

for every $C_x, C_y \in X/I$. This completes the proof.

THEOREM 8. Let $X$ be a positive implicative BCK-algebra. Then every left regular map $\alpha$ of $X$ has a fixed point, i.e., $\alpha(y) = y$ for some $y \in X$.

Proof. Note that $X$ satisfies the identities

(6) $(x \ast (y \ast x)) \ast (x \ast y) = (y \ast (y \ast x)) \ast (x \ast y)$,
(7) $x \ast 0 = x$.

Hence we have

$$(\alpha(x) \ast (x \ast \alpha(x))) \ast (\alpha(x) \ast x) = (x \ast (x \ast \alpha(x))) \ast (\alpha(x) \ast x)$$

for each $x \in X$. It follows from $\alpha(x) \leq x$ and (7) that

$$\alpha(x \ast (x \ast \alpha(x))) = \alpha(x) \ast (x \ast \alpha(x)) = x \ast (x \ast \alpha(x))$$

for all $x \in X$, which completes the proof.

References


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