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GENERAL CONCEPTS OF REGULARITY OF NEAR RINGS

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1.Introduction

The concepts of regularity of near-rings have been studied by J. C. Beidleman [2] and S. Ligh [6]. In 198O G. Mason [7] introduced the notions of strong regularity of near-rings, and he proved that for any zero-symmetric near-ring with identity, the concepts of left regularity, strong left regularity and strong right regularity of near-rings are equivalent.

In 1985 M. Ōhori [8], and in 1991 Y. Hirano [4] investigated the characterization of strong π-regularity of ring, and of π-regularity of rings.

In this paper, we will introduce more general concepts π-regularity of near-rings that is $\mathcal{K}$-regularity and semi π-regularity and we shall characterize relationships between them. We see that every left and right π-regularity of near-ring is π-regularity. We can prove that, using some Lemmas, for zero-symmetric near-ring with left identity, the notions of regularity, π-regularity, left $\mathcal{K}$-regularity and left semi π-regularity of near-rings are equivalent.

C. Faith [3] studied chain conditions on principal annihilator left ideals and on principal left ideals of ring with identity, we will study chain conditions on principal annihilator left ideals and on principal N-subgroups of near-rings.

We will generalize the concepts of bipotent near-rings which are called the generalized left bipotent near-ring and the generalized right bipotent near-ring and we will investigate relationships between left $\mathcal{K}$-regularity and DCCN for principal N-subgroups and generalized left bipotent near-rings, between right $\mathcal{K}$-regularity and ACCL for principal annihilator left ideals and generalized right bipotent near-rings.

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2. Preliminaries

A near-ring N is an algebraic system $(N,+)$ such that $(N,+,\cdot)$ is a group (not necessarily abelian) and $(N,\cdot)$ is a semigroup with right distributive laws hold for any three elements in N. In general, for any element a in N, $a0=0$ is not true, and $N_0=\{a\in N : a0=0\}$ is a subnear-ring of N, if $N=N_0$ then N is called a zero-symmetric near-ring and other basic definitions in near-rings and N-groups see the text book of Gunter Pilz [9] as " Near-Ring ".

A near-ring N is called π-regular if for any element a in N there exists a positive integer n such that a^n is a regular element, and left π-regular if for any element a in N there exists a positive integer n such that a^n is a left regular element, and right π-regular if for any element a in N there exists a positive integer n such that a^n is a right regular element.

A near-ring N is called left S-unital if for any element a in N, a is contained in Na. Similarly for right S-unital near-ring.

For any a in N, Na is called a principal (left) N-subgroups of N. We see that $Na \supset Na^2 \supset Na^3 \supset \cdots$ descending chain of principal N-subgroups of N, denoted that $L(a)$ is a (left) annifilator of a in N, we called that $L(a)$ is a principal annihilator left ideal of N, clearly we obtain that $L(a)\subset L(a^2)\subset L(a^3)\cdots$ ascending chain of principal annihilator left ideals of N.

LEMMA 2.1. *Let N be a left (or right) regular near-ring. If for any a,b in N such that $ab=0$ then $(ba)^n=b0$, for all positive integer n.*

LEMMA 2.2. *Let N be a left (or right) regular near-ring. If for each a,b in N with $ab=0$ and $a^n=a0$, for all positive integer n which are greater than or equal to 2, then $a=0$. In this case, if $N=N_0$ then N is reduced.*

LEMMA 2.3. (G. MASON [7]). *Let N be a zero-symmetric, reduced near-ring. If for any a,b in N such that $ab=0$, then $ba=0$.*

Proof. Clearly, from Lemma 2.1.

LEMMA 2.4. *Let N be a zero-symmetric near-ring with left identity. If N is reduced, then every idempotent is central.*

Proof. Let e be an idempotent element in N and let x in N. Then $(ex-exe)e=0$. By Lemma 2.3, $e(ex-exe)=0$. On the other hand, since $(ex-exe)ex=0$, $ex(ex-exe)-exe(ex-exe)=0$, by hypothesis, $ex=exe$.

Next, if N has a left identity 1, then, since $(1-e)e=0$ by Lemma 2.3, $e(1-e)=0$ and since $(xe-exe)e=xe-exe, (1-e)xe=xe-exe$, we obtain that

$$(xe-exe)^2=(xe-exe)e(1-e)xe=0.$$

Then, since N is reduced, $xe=exe$. Hence $ex=xe$. Consequently e is central.

3. More general concepts of regularity and their relationships

A near-ring N is said to be left semi π-regular if for any element a in N there exists an element x in N such that $a^n=axa^n$ for some positive integer n, and right semi π-regular if for any element a in N there exists an element x in N such that $a^n=axa^n$ for some positive integer n.

N is called left $\mathcal{K}$-regular if for every element a in N, there exists an element x in N such that $a^n=xa^{n+1}$ for some positive integer n, and right $\mathcal{K}$-regular if for every element a in N, there exists an element x in N such that $a^n=a^{n+1}x$ for some positive integer n.

In general, every π-regular near-ring is left semi π-regular and right semi π-regular, every left π-regular near-ring is left $\mathcal{K}$-regular and every left $\mathcal{K}$-regular near-ring is left semi π-regular but not conversely.

Simillarly for right π-regular, right $\mathcal{K}$-regular and right semi π-regular.

There exist many examples of semi π-regularity and $\mathcal{K}$-regularity of near-rings, we can easily show for finite near-rings. See near-rings of low order in appendix of [9].

PROPOSITION 3.1.

(1) *Let N be a left S-unital near-ring. Then N is left regular if and only if N is left bipotent.*
(2) *Let N be a right S-unital near-ring. Then N is right regular if and only if N is right bipotent.*

Proof. These are easy, so left to the readers. See, Remark 4.7. in this paper, section 4.

PROPOSITION 3.2. *If a near-ring N is left semi π-regular and left regular, then N is regular.*

Proof. Since N is left semi π-regular, for each a in N, there exist an element x in N such that $a^n = axa^n$, for some positive integer n. Since $(a^{n-1} - axa^{n-1})a = 0$ by Lemma 2.1, $a(a^{n-1} - axa^{n-1}) = a0$.

Now, we see that

$$\begin{aligned}(a^{n-1} - axa^{n-1})^2 &= a^{n-1}(a^{n-1} - axa^{n-1}) - axa^{n-1}(a^{n-1} - axa^{n-1}) \\ &= (a^{n-1} - axa^{n-1})0.\end{aligned}$$

By Lemma 2.2, $a^{n-1} = axa^{n-1}$. After $n-2$ steps from this equality, we obtain that, $a = axa$. Hence N is regular.

PROPOSITION 3.3. *Let N is zero-symmetric with left identity and reduced. The following statements are equivalent :*

(1) *N is π-regular.*
(2) *N is left $\mathcal{K}$-regular.*
(3) *N is left semi π-regular.*
(4) *N is regular.*

Proof. (1) $\Rightarrow$ (2). Suppose N is π-regular. Then for any element a in N there exists an element x in N such that $a^n = a^n x a^n$, for some positive integer n. Since xa^n is an idempotent, from Lemma 2.4, it is central. Thus we have

$$a^n = a^n x a^n = x a^n a^n = x a^{n-1} a^{n+1} = y a^{n+1},$$

where y is denoted by xa^{n-1}. Hence N is left $\mathcal{K}$-regular.

(2) $\Rightarrow$ (3). It follows from the beginning part in this section.

(3) $\Rightarrow$ (4). Assume that N is left semi π-regular. Then for each a in N, there exists an element x in N such that $a^n = axa^n$ for some positive integer n. Since $(a^{n-1} - axa^{n-1})a = 0$, by Lemma 2.3, $a(a^{n-1} - axa^{n-1}) = 0$. Now we see that

$$(a^{n-1} - axa^{n-1})^2 = a^{n-1}(a^{n-1} - axa^{n-1}) - axa^{n-1}(a^{n-1} - axa^{n-1}) = 0.$$

Since N is reduced, $a^{n-1} = axa^{n-1}$. Continuing this procedure to $n-2$ steps, we obtain that $a = axa$ that is N is regular.

(4) $\Rightarrow$ (1). Obvious.

PROPOSITION 3.4. *If a near-ring N is left N and π-regular then N is right $\mathcal{K}$-regular.*

Proof. Using Lemma 2.1 and Lemma 2.2.

COROLLARY 3.5. *Let N is zero-symmetric with left identity and left regular. The following statements are equivalent :*

(1) N is regular.
(2) N *is π-regular.*
(3) N *is left $\mathcal{K}$-regular.*
(4) N *is left semi π-regular.*

PROPOSITION 3.6. *If N is a semi π-regular near-ring with the property that for each non-zero element a in N, there is a unique element x in N and some positive integer n such that $a^n = axa^n$ (or $a^n = a^nxa$), then N is integral, in particular N is reduced. Furthermore, every non zero idempotent is a right identity.*

Proof. We will prove the first statement and the remainder part is easily proved.

Let a be any non zero element in N. If $ba = 0$ for all b in N. Then there is a positive integer n and a unique x in N such that $a^n = axa^n$ and we see that $a(x+b)a^n = axa^n$. By uniqueness property, we obtain that $x + b = x$. Consequently, $b = 0$. Therefore N is integral.

COROLLARY 3.7. *If N is a regular near-ring with the property : for any non zero element a in N, there exists a unique x in N such that $a = axa$, then N is integral, in particular N is reduced. Moreover, every non zero idempotent is a right identity.*

These statements in corollary 3.7. are very important in ring theory and semigroup theory (: inverse semigroup).

4. Generalized bipotent near-rings and their application

A near-ring N is said to be generalized left bipotent (: GLB), if for any element a in N, there exists a positive integer n, such that $Na^n = Na^{n+1}$.

Similarly, we can also define generalized right bipotent (: GRB).

A near-ring N satisfies DCC. on principal left N-subgroups of N, if for each element a in N there is a positive integer n such that

$Na^n = Na^{n+1} = Na^{n+2} = \cdots$. A near-ring N has ACC. on principal annihilator left ideals of N if for every element a in N there is a positive integer n such $L(a^n) = L(a^{n+1}) = L(a^{n+2}) = \cdots$. Obviously, every left bipotent near-ring is GLB and every right bipotent near-ring is also GRB.

PROPOSITION 4.1. *Let N be a GLB near-ring. If there exists an element in N which is not a (right) zero divisor. Then N is monogenic and left bipotent.*

Proof. Let a in N which is not a zero divisor. Then a^n is also not a zero divisor for any positive integer n. Indeed, if a^n is a zero divisor for some positive integer n. Then there exists non-zero element x in N such that $xa^n = 0$.

Now, $xa^{n-1}a = 0$ implies that $xa^{n-1} = 0$ because a is not a zero divisor. Continuing this process, we have $x = 0$ which is contradiction.

Next, since N is GLB, there exists a positive integer m, such that $Na^m = Na^{m+1}$. Let x be any element of N. Then there is an element y in N such that $xa^m = ya^{m+1}$. This implies that $(x - ya)a^m = 0$. Since a^m is not a zero divisor, $x = ya$, we see that $Na = N$. Thus N is monogenic. Clearly, N is left bipotent.

COROLLARY 4.2. *Let N be a near-ring with DCC. on principal left N-subgroups of N. If there exists an element in N that is not a (right) zero divisor. Then N is monogenic and left bipotent.*

REMARK 4.3. In proposition 4.1, Corollary 4.2, if $N = N_d$, and "left" is replaced by "right" and " right " is replaced by "left", then N is also monogenic (:right, in the sense) and right bipotent.

THEOREM 4.4. *Let N be any left S-unital near-ring. Then the following statements are equivalent :*

(1) *N is left $\mathcal{K}$-regular.*
(2) *N has the DCC. on principal left N-subgroups of N.*
(3) *N has the almost DCC. on principal left N-subgroups of N.*
(4) *N is GLB.*

Proof. (1) $\Rightarrow$ (2). Assume N is left $\mathcal{K}$-regular. Let a be any element in N. Since N is left S-unital, Na is a principal left N-subgroup of N

generated by a. Consider

$$Na \supset Na^2 \supset Na^3 \supset \cdots$$

be descending chain ofthe principal left N-subgroups of N. Since N is left $\mathcal{K}$-regular, there exists an element x in N and exists a positive integer n, such that $a^n = xa^{n+1}$. Now

$$Na^n = Nxa^{n+1} \subset Na^{n+1} = Naa^n \subset Na^n.$$

Thus we obtain that $Na^n = Na^{n+1}$. Similarly, we have that

$$Na^{n+1} = Na^{n+2} = Na^{n+3} = \cdots$$

Hence N satisfies the DCC. on principal left N-subgroups.

(2) $\Rightarrow$ (3). For almost DCC. or almost ACC. concepts, see the paper [11]. Almost DCC.(Almost ACC. resp.) concept is more general concept of DCC.(ACC. resp.) For S-unital near-ring (or ring), " almost DCC.(almost ACC. resp.) on N-subgroups (or ideals) " is equivalent to " DCC.(ACC. resp.) on N-subgroups (or ideals) ".

(3) $\Rightarrow$ (4). From the fact that " GLB " is a general concept of " DCC. on principal left N-subgroups of N ".

(4) $\Rightarrow$ (1). Suppose N is GLB. Let a be any element of N. Then there exists a positive integer n such that $Na^n = Na^{n+1}$. Since N is left S-unital, a^n is an element of Na^n so a^n is also an element of Na^{n+1}. Then there exists x in N such that $a^n = xa^{n+1}$. Hence N is left $\mathcal{K}$-regular.

Corollary 4.5. *let N be any near-ring with left identity. Then the following statements are equivalent :*

(1) *N is left $\mathcal{K}$-regular.*
(2) *N has the DCC. on principal left N-subgroups of N.*
(3) *N has the almost DCC. on principal left N-subgroups of N.*
(4) *N is GLB.*

Theorem 4.6. *Let N be any right S-unital near-ring. Then the following statements are equivalent :*

(1) *N is right $\mathcal{K}$-regular.*
(2) *N has the DCC. on principal right N-subgroups of N.*
(3) *N has the almost DCC. on principal right N-subgroups of N.*
(4) *N is GRB.*

Proof. Similarly, for the proof of proposition 4.4.

REMARK 4.7. Proposition 3.1 (1) and (2) are special cases of Theorem 4.4 and Theorem 4.6.

LEMMA 4.8. *Let N be any near-ring. If N is right $\mathcal{K}$-regular then N satisfies the ACC. on principal annihilator left ideals of N.*

Proof. Straightforwards.

THEOREM 4.9. *Let N be a zero symmetric left $\mathcal{K}$-regular near-ring with left identity. Then the following conditions are equivalent :*

(1) *N is right $\mathcal{K}$-regular.*
(2) *N has the ACC. on principal annihilator left ideals of N.*
(3) *N has the almost ACC. on principal annihilator left ideals of N.*

Proof. (1) $\Rightarrow$ (2). From Lemma 4.8.
(2) $\Rightarrow$ (3). Using zero symmetric, it is straightforward.
(3) $\Rightarrow$ (1). For any a in N, consider $L(a^m) = L(a^m+1) = \cdots$ for some positive integer m, since N is left $\mathcal{K}$-regular, there is an element x in N and some positive integer t such that $a^t = xa^{t+1}$ without loss of generality, we select $n := m = t$. So we have $L(a^n) = L(a^{n+1}) = \cdots$, and $a^n = xa^{n+1}$ for some positive integer n. Then for all positive integer k, $x^k a^n = x^{k+1}a^{k+1}$.
But, $a^{n+1} = aa^n = axa^{n+1}$ implies $(1-ax)a^{n+1} = 0$, that is $(1-ax)$ is an element of $L(a^{n+1}) = L(a^n)$. Hence $a^n = axa^n$.
Since

$$a^{n+1} = aa^n = aaxa^n = a^2x(xa^{n+1}) = a^2x^2a^{n+1},$$

$(1-a^2x^2)a^{n+1} = 0$ implies $(1-a^2x^2)$ is an element of $L(a^n)$. Thus we have $(1-a^2x^2)a^n = 0$, that is $a^n = a^2x^2a^2$. Continuing this process ($n+1$) steps, we obtain the equation $a^n = a^{n+1}x^{n+1}a^n$. Putting $x^{n+1}a^n$ as y in N then $a^n = a^{n+}1y$. Consequently N is a right $\mathcal{K}$-regular near-ring.

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