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ON NEW CRITERIA FOR MEROMORPHIC P-VALENT CONVEX FUNCTIONS

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1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \frac{a_0}{z^{p-1}} + \cdots + a_{k+p-1}z^k + \cdots \tag{1.1}$$

which are regular in the annulus $D = \{z : 0 < |z| < 1\}$, where p is a positive integer. The Hadamard product or convolution of two functions f, g in $\sum_p$ will be denoted by $f * g$. Let

(1.2)

$$\begin{aligned} D^{n+p-1}f(z) &= \frac{1}{z^p(1-z)^{n+p}} * f(z), \qquad (z \in D) \\ &= \frac{1}{z^p}\left(\frac{z^{n+2p-1}f(z)}{(n+p-1)!}\right)^{(n+p-1)} \\ &= \frac{1}{z^p} + (n+p)a_0\frac{1}{z^{p-1}} + \frac{(n+p+1)(n+p)}{2!}a_1\frac{1}{z^{p-2}} + \cdots \\ &\quad \cdots + \frac{(n+k+2p-1)\cdots(n+p)}{(k+p)!}a_{k+p-1}z^k + \cdots, \end{aligned}$$

where n is any integer greater than $-p$.

In this paper, among other things, we shall show that a function $f(z)$ in $\sum_p$, which satisfies one of the conditions

$$Re\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} - (p+1)\right\} < -p\frac{n+p-1}{n+p}, \quad (z \in D), \tag{1.3}$$

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where n is any integer greater than $-p$, is meromorphic p-valent convex in D. More precisely, it is proved that for the classes J_{n+p-1} of functions in $\sum_p$ satisfying (1.3)

$$J_{n+p} \subset J_{n+p-1} \tag{1.4}$$

holds. Since J_0 equals $\sum_k$ (the class of meromorphic p-valent convex functions), the convexity of members of J_{n+p-1} is a consequence of (1.4).

REMARK. When $p = 1$, J_{n+p-1} reduces to the class of meromorphic convex functions of Uralegaddi and Ganigi [8].

2. The classes J_{n+p-1}

Now we need the following lemma due to I.S. Jack [3].

LEMMA. *Let w be non-constant regular in $U = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \geq 1$.*

THEOREM 1. *$J_{n+p} \subset J_{n+p-1}$ for each integer n greater than $-p$.*

Proof. Let $f(z) \in J_{n+p}$. Then

$$Re\left\{\frac{(D^{n+p+1}f(z))'}{(D^{n+p}f(z))'} - (p+1)\right\} < -p\frac{n+p}{n+p+1}. \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$Re\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} - (p+1)\right\} < -p\frac{n+p-1}{n+p}. \tag{2.2}$$

Define $w(z)$ in $U = \{z : |z| < 1\}$ by

$$\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} - (p+1) = -p\left\{\frac{n+p-1}{n+p} + \frac{1}{n+p}\frac{1-w(z)}{1+w(z)}\right\}. \tag{2.3}$$

Clearly $w(z)$ is regular and $w(0) = 0$. Equation (2.3) may be written as

$$\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} = \frac{n+p+(n+3p)w(z)}{(n+p)(1+w(z))}. \tag{2.4}$$

Differentiating (2.4) logarithmically, we obtain

(2.5)

$$\frac{z(D^{n+p}f(z))''}{(D^{n+p}f(z))'} - \frac{z(D^{n+p-1}f(z))''}{(D^{n+p-1}f(z))'} = \frac{2pzw'(z)}{(1+w(z))(n+p+(n+3p)w(z))}.$$

From the following identity

(2.6) $z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - (n+2p)D^{n+p-1}f(z)$,

(2.7)

$z(D^{n+p-1}f(z))'' = (n+p)(D^{n+p}f(z))' - (n+2p+1)(D^{n+p-1}f(z))'$.

Using the identity (2.7), equation (2.5) may be written as

(2.8)

$$\frac{(D^{n+p+1}f(z))'}{(D^{n+p}f(z))'} - (p+1) + p\frac{n+p}{n+p+1} = -\frac{2p+n}{n+p+1} + \frac{1}{n+p+1}\left[\frac{n+p+(n+3p)w(z)}{1+w(z)} + \frac{2pzw'(z)}{(1+w(z))(n+p+(n+3p)w(z))}\right].$$

That is,

(2.9)

$$\frac{(D^{n+p+1}f(z))'}{(D^{n+p}f(z))'} - (p+1) + p\frac{n}{n+p}n+p+1 = \frac{1}{n+p+1}\left[-p\frac{1-w(z)}{1+w(z)} + \frac{2pzw'(z)}{(1+w(z))(n+p+(n+3p)w(z))}\right].$$

We claim that $|w(z)| < 1$ in U. For otherwise (by Jack's lemma) there exists z_0, $|z| < 1$ such that

(2.10) $$z_0w'(z_0) = kw(z_0),$$

where $|w(z_0)| = 1$ and $k \geq 1$. (2.9) in conjuction with (2.10) yields

(2.11)
$$\begin{aligned}\frac{(D^{n+p+1}f(z_0))'}{(D^{n+p}f(z_0))'} - (p+1) + p\frac{n+p}{n+p+1} \\ = \frac{1}{n+p+1}\Bigg[-p\frac{1-w(z_0)}{1+w(z_0)} \\ +\frac{2pkw(z_0)}{(1+w(z_0))(n+p+(n+3p)w(z_0))}\Bigg].\end{aligned}$$

Thus

(2.12)
$$\begin{aligned}Re\left\{\frac{(D^{n+p+1}f(z_0))'}{(D^{n+p}f(z_0))'} - (p+1) + p\frac{n+p}{n+p+1}\right\} \\ \geq \frac{p}{(n+p+1)(2n+4p)} \geq 0\end{aligned}$$

which contradicts (2.1). Hence $|w(z)| < 1$ in U and from (2.3) it follows that $f(z) \in J_{n+p-1}$.

THEOREM 2. *Let $f(z) \in \sum_p$ and for a given integer $n > -p$ and $c > 0$, satisfy the condition*

$$\begin{aligned}Re\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} - (p+1)\right\} \\ < \frac{p(1-2(c+p)(n+p-1))}{2(n+p)(c+p)}, \qquad (z \in U).\end{aligned} \tag{2.13}$$

Then

$$F(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \tag{2.14}$$

belongs to J_{n+p-1}.

Proof. Using the identities

$$z(D^{n+p-1}F(z))' = cD^{n+p-1}f(z) - (c+p)D^{n+p-1}F(z) \tag{2.15}$$

and

(2.16)
$$z(D^{n+p-1}F(z))' = (n+p)D^{n+p}F(z) - (n+2p)D^{n+p-1}F(z),$$

the condition (2.13) may be written as

(2.17)
$$Re\left\{\frac{(n+p+1)\frac{(D^{n+p+1}F(z))'}{(D^{n+p}F(z))'}(n+p+1-c)}{(n+p)-(n+p-c)\frac{(D^{n+p-1}F(z))'}{(D^{n+p}F(z))'}} - (p+1)\right\}$$
$$< \frac{p(1-2(c+p)(n+p-1))}{2(n+p)(c+p)}.$$

We have to prove that (2.17) implies the inequality

$$Re\left\{\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'} - (p+1)\right\} < -p\frac{n+p-1}{n+p}. \tag{2.18}$$

Define $w(z)$ in U by

(2.19)
$$\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'} - (p+1)$$
$$= -p\left\{\frac{n+p-1}{n+p} + \frac{1}{n+p}\frac{1-w(z)}{1+w(z)}\right\}.$$

Clearly $w(z)$ is regular and $w(0) = 0$. The equation (2.19) may be written as

$$\frac{(D^{n+p}F(z))'}{(D^{n+p-1}F(z))'} = \frac{n+p+(n+3p)w(z)}{(n+p)(1+w(z))}. \tag{2.20}$$

Also from (2.16) we have

(2.21)
$$z(D^{n+p-1}F(z))'' = (n+p)(D^{n+p}F(z))'$$
$$- (n+2p+1)(D^{n+p-1}F(z))'.$$

Differentiating (2.20) logarithmically and using the identity (2.21), after simple computation we obtain

(2.22)

$$\frac{(n+p+1)\frac{(D^{n+p+1}F(z))'}{(D^{n+p}F(z))'}-(n+p+1-c)}{(n+p)-(n+p-c)\frac{(D^{n+p-1}F(z))'}{(D^{n+p}F(z))'}}-(p+1)$$
$$=-\frac{p(n+p-1)}{n+p}-\frac{p}{n+p}\frac{1-w(z)}{1+w(z)}$$
$$+\frac{2pzw'(z)}{(n+p)(1+w(z))(c+(2p+c)w(z))}.$$

We claim that $|w(z)|<1$ in U. For otherwise (by Jack's lemma) there exists z_0,$|z|<1$ such that

$$z_0w'(z)=kw(z_0) \tag{2.23}$$

where $|w(z_0)|=1$ and $k\geq 1$. Combining (2.22) and (2.23), we obtain

(2.24)

$$\frac{(n+p+1)\frac{(D^{n+p+1}F(z_0))'}{(D^{n+p}F(z_0))'}-(n+p+1-c)}{(n+p)-(n+p-c)\frac{(D^{n+p-1}F(z_0))'}{(D^{n+p}F(z_0))'}}-(p+1)$$
$$=-\frac{p(n+p-1)}{n+p}-\frac{p}{n+p}\frac{1-w(z_0)}{1+w(z_0)}$$
$$+\frac{2pkw(z_0)}{(n+p)(1+w(z_0))(c+(2p+c)w(z_0))}.$$

Thus

(2.25)

$$Re\left\{\frac{(n+p+1)\frac{(D^{n+p+1}F(z_0))'}{(D^{n+p}F(z_0))'}-(n+p+1-c)}{(n+p)-(n+p-c)\frac{(D^{n+p-1}F(z_0))'}{(D^{n+p}F(z_0))'}}-(p+1)\right\}$$
$$\geq\frac{p(1-2(c+p)(n+p-1))}{2(n+p)(c+p)},$$

which contradicts (2.13). Hence $|w(z)|<1$ in U and from (2.19) it follows that $F(z)\in J_{n+p-1}$.

Putting $n=-p+1$ in the statement of Theorem 2, we have the following

COROLLARY. *If* $f(z) \in \sum_p$ *and satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{p}{2(c+p)}, \tag{2.26}$$

then

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt \tag{2.27}$$

belongs to $\sum_k$.

THEOREM 3. *If* $f(z) \in J_{n+p-1}$, *then*

$$F(z) = \frac{n+p}{z^{n+2p}} \int_0^z t^{n+2p-1} f(t)dt \tag{2.28}$$

belongs to J_{n+p}.

Proof. For

$$F(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt, \tag{2.29}$$

we have

$$\begin{aligned} cD^{n+p-1}f(z) = (n+p)D^{n+p}F(z) \\ -(n+p-c)D^{n+p-1}F(z) \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} cD^{n+p}f(z) = (n+p+1)D^{n+p+1}F(z) \\ -(n+p+1-c)D^{n+p}F(z). \end{aligned} \tag{2.31}$$

Taking $c = n+p$ in the above relations, we obtain

$$\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'} = \frac{(n+p+1)(D^{n+p+1}F(z))' - (D^{n+p}F(z))'}{(n+p)(D^{n+p}F(z))'}, \tag{2.32}$$

which reduces to

$$(2.33)\qquad \frac{(n+p+1)(D^{n+p+1}F(z))'}{(n+p)(D^{n+p}F(z))'}-\frac{1}{n+p}=\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'}.$$

Thus

(2.34)

$$Re\left\{\frac{(n+p+1)(D^{n+p+1}F(z))'}{(n+p)(D^{n+p}F(z))'}-\frac{1}{n+p}-(p+1)\right\}$$
$$=Re\left\{\frac{(D^{n+p}f(z))'}{(D^{n+p-1}f(z))'}-(p+1)\right\}<-p\frac{n+p-1}{n+p}.$$

From which it follows that

$$(2.35)\qquad Re\left\{\frac{(D^{n+p+1}F(z))'}{(D^{n+p}F(z))'}-(p+1)\right\}<-p\frac{n+p}{n+p+1}.$$

This completes the proof of theorem.

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