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ON SEQUENTIAL HAUSDORFF COMPACTIFICATION SATISFYING THE FIRST AXIOM OF COUNTABILITY

SEONHEE KWEON AND WOO CHORL HONG

It is well known that any metrizable space admits a sequential Housdorff compactification [2]. The purpose of this paper is to obtain a condition for a Hausdorff space satisfying the first axiom of countability to have a sequential Hausdorff compactification satisfying the first axiom of countability. Some steps of the construction of sequential Hausdorff compactifications in this paper are inspired by Wu [4]. All spaces in this paper are assumed to be Hausdorff satisfying the first axiom of countability. Let X be a topological space, $\{X_p \mid p = 1, 2, \cdots, k\}$ a finite family of spaces, and $\mathcal{A} = \{f_p | f_p : X \longrightarrow X_p$ is continuous, $p = 1, 2, \cdots, k\}$. We call a sequence (x_n) in X an $\mathcal{A}$-sequence in X if $(f_p(x_n))$ has a cluster point, for each $p = 1, 2, \cdots, k$.

PROPOSITION 1. *Let X is sequentially compact if and only if*

(1) *$f_p(X)$ is contained in a sequential compact subset C_{f_p} for each $f_p \in \mathcal{A}$,* and
(2) *every $\mathcal{A}$-sequence in X has a cluster point in X.*

Proof. Necessity is obvious. To prove sufficiency, let (x_n) be a sequence in X. Then, by (1), $(f_p(x_n))$ has a cluster point in X_p for each $f_p \in \mathcal{A}$, and so, (x_n) is an $\mathcal{A}$-sequence in X. Thus, (x_n) has a cluster point in X by (2).

LEMMA. *Let E and F be subspaces of X with $E \subset F \subset Cl_X(E)$, where $Cl_X(E)$ is the closure of E in X. If X has the weak topology induced by $\mathcal{A}$, then the following statements are equivalent.*

(1) *Every $\mathcal{A}$-sequence in E has a cluster point in F.*
(2) *Every $\mathcal{A}$-sequence in F has a cluster point in F.*

Proof. To prove (1) $\Rightarrow$ (2), let (y_n) be an $\mathcal{A}$-sequence in F. Then $(f_p(y_n))$ has a cluster point z_p in X_p, for each $f_p \in \mathcal{A}$. So there

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exists a subsequence $(f_p(y_{p(i)}))$ of $(f_p(y_n))$ such that $(f_p(y_{p(i)}))$ converges to z_p, for each $f_p \in \mathcal{A}$. Since $F \subset \mathrm{Cl}_X(E)$, for each $y_{p(i)}$, there exists a sequence $(x^m_{p(i)})$ in E such that $(x^m_{p(i)})$ converges to $y_{p(i)}$. By continuity of f_p, $(f_p(x^m_{p(i)}))$ converges to $f_p(y_{p(i)})$. Let $B_p = \{f_p(x^m_{p(i)}) \mid m, i \in N\}$. Then $f_p(y_{p(i)}) \in \mathrm{Cl}_{X_p}(B_p)$, for each i, and hence we have $z_p \in \mathrm{Cl}_{X_p}(B_p)$. Thus there exists a sequence $(f_p(x^n_p))$ in B_p such that $(f_p(x^n_p))$ converges to z_p, for each $p = 1, 2, \cdots, k$. Let (l_r) be the sequence such that $l_1 = x^1_1, l_2 = x^1_2, \cdots, l_k = x^1_k$, $l_{k+1} = x^2_1, \cdots, l_{2k} = x^2_k, \cdots$. Then (l_r) is a sequence in E and we see that $(f_p(l_r))$ has a cluster point z_p, for each $f_p \in \mathcal{A}$. So (l_r) is an $\mathcal{A}$-sequence in E. By (1), (l_r) has a cluster point x in F. Then it is easily verified that for some p_0, there exists a subsequence $(x^{n(j)}_{p_0})$ of $(x^n_{p_0})$ such that $(x^{n(j)}_{p_0})$ converges to x. It follows that $(f_{p_0}(x^{n(j)}_{p_0}))$ converges to $(f_{p_0}(x))$. Since $(f_{p_0}(x^n_{p_0}))$ converges to z_{p_0} and $(f_{p_0}(x^{n(j)}_{p_0}))$ is a subsequence of $(f_{p_0}(x^n_{p_0})), (f_{p_0}(x^{n(j)}_{p_0}))$ converges to z_{p_0}. By Hausdorffness of X_{p_0}, we have $f_{p_0}(x) = z_{p_0}$. Consequently, we have that $(f_{p_0}(y_{p_0(i)}))$ converges to $f_{p_0}(x)$. Since X has the weak topology induced by $\mathcal{A}$, $(y_{p_0(i)})$ converges to x. Therefore, (y_n) has a cluster point x in F.

The converse is trivial, and hence the proof is complete.

PROPOSITION 2. *Let E and F be subspaces of X with $E \subset F \subset Cl_X(E)$. If X has the weak topology induced by $\mathcal{A}$, then F is sequentially compact if and only if*

(1) *$Cl_{X_p}(f_p(E))$ is sequentially compact for each $f_p \in \mathcal{A}$, and*
(2) *every $\mathcal{A}$-sequence in E has a cluster point in F.*

Proof. We assume that F is sequentially compact. Then $f_p(F)$ is sequentially compact by continuity of f_p, for each $f_p \in \mathcal{A}$. Since every sequential compact subset of a Hausdorff space satisfying the first axiom of countability is closed (See [1], p.230),

$$\mathrm{Cl}_{X_p}(f_p(E)) \subset \mathrm{Cl}_{X_p}(f_p(F)) = f_p(F).$$

And since every closed subspace of a sequential compact space is sequentially compact (See [1], p.230), $\mathrm{Cl}_{X_p}(f_p(E))$ is sequentially compact for each $f_p \in \mathcal{A}$. Thus (1) is proved. (2) is followed by Proposition 1 and Lemma.

Conversely, since

$$f_p(F) \subset f_p(\mathrm{Cl}_X(E)) \subset \mathrm{Cl}_{X_p}(f_p(E)),$$

$f_p(F)$ is contained in a sequential compact subset $\mathrm{Cl}_{X_p}(f_p(E))$ by (1), for each $f_p \in \mathcal{A}$. By (2) and Lemma, every $\mathcal{A}$-sequence in F has a cluster point in F. Thus F is sequentially compact by Proposition 1.

Let $S = \{(x_n) \mid (x_n)$ is a sequence in X such that $(f_p(x_n))$ converges in X_p, for each $f_p \in \mathcal{A}\}$. Define a relation $\simeq$ on S by $(x_n) \simeq (y_n)$ if and only if $\lim(f_p(x_n)) = \lim(f_p(y_n))$ for each $f_p \in \mathcal{A}$. It is trivial that this relation $\simeq$ is an equivalence relation on S. Let $X^*(\mathcal{A})$ be the quotient set of S by $\simeq$, and $[(x_n)] \in X^*(\mathcal{A})$ the equivalence class containing (x_n). And, for each $f_p \in \mathcal{A}$, define a mapping f_p^* on $X^*(\mathcal{A})$ into X_p by $f_p^*([(x_n)]) = \lim(f_p(x_n))$ for each $f_p \in \mathcal{A}$. Then, clearly, f_p^* is well-defined and injective. Now we denote by $\mathcal{A}^* = \{f_p^* \mid f_p \in \mathcal{A}\}$. Hereafter, we assume that $X^*(\mathcal{A})$ has the weak topology induced by $\mathcal{A}^*$. Then we call a sequence (α_n) in $X^*(\mathcal{A})$ an $\mathcal{A}^*$-sequence in $X^*(\mathcal{A})$ if $(f_p^*(\alpha_n))$ has a cluster point in X_p, for each $f_p^* \in \mathcal{A}^*$. Then we have the following:

PROPOSITION 3. *$\mathcal{A}^*$ is a Hausdorff space satisfying the first axiom of countability.*

Proof. We first show that $X^*(\mathcal{A})$ is Hausdorff. Let $\alpha \neq \beta$ in $X^*(\mathcal{A})$ Then $f_p^*(\alpha) \neq f_p^*(\beta)$ for some $f_p^* \in \mathcal{A}^*$. Since X_p is Hausdorff, there are disjoint open nbds U and V of $f_p^*(\alpha)$ and $f_p^*(\beta)$ in X_p, respectively. Hence we have disjoint open nbds ${f_p^*}^{-1}(U)$ and ${f_p^*}^{-1}(V)$ of α and β in $X^*(\mathcal{A})$, respectively.

To prove that $X^*(\mathcal{A})$ satisfies the first axiom of countability, let $\alpha \in X^*(\mathcal{A})$. Then $f_p^*(\alpha) \in X_p$ for each $f_p^* \in \mathcal{A}^*$. Since X_p satisfies the first axiom of countabality for each p, there is a countable local base $\mathcal{B}_p$ of $f_p^*(\alpha)$ for each p. Then, let $\mathcal{B} = \cup_{p=1}^k \{{f_p^*}^{-1}(U_p) \mid U_p \in \mathcal{B}_p\}, \mathcal{B}$ forms a countable local base of α in $X^*(\mathcal{A})$. For, let V be an open nbd of α. Since $X^*(\mathcal{A})$ has the weak topology induced by $\mathcal{A}^*$, there is an open nbd V_p of $f_p^*(\alpha)$ such that ${f_p^*}^{-1}(V_p) \subset V$. So, there is an element U_p of $\mathcal{B}_p$ with $f_p^*(\alpha) \in U_p \subset V_p$. It follows that $\alpha \in {f_p^*}^{-1}(U_p) \subset {f_p^*}^{-1}(V_p) \subset V$. Hence $X^*(\mathcal{A})$ satisfies the first axiom of countability.

Now we define a mapping $e : X \longrightarrow X^*(\mathcal{A})$ by $e(x) = [(x)]$ for each $x \in X$, where (x) is the constant sequence in X whose n-th term is x for all indices $n \in N$. Then, clearly, e is injective.

PROPOSITION 4. *e is a dense embedding if and only if X has the weak topology induced by $\mathcal{A}$.*

Proof. If e is a dense embedding, then $e(X)$ is homeomorphic to X, and hence for each open set U in X, $e(U) = e(X) \cap V_U$ for some open set V_U in $X^*(\mathcal{A})$. Since $X^*(\mathcal{A})$ has the weak topology induced by $\mathcal{A}^*$, we can represent V_U as the union of some basic open sets in $X^*(\mathcal{A})$, let $V_U = \cup_{\iota\in I} {f^*_{p_\iota}}^{-1}(U_{p_\iota})$, where $p_\iota = 1, 2, \cdots k$ and U_{p_ι} is open in X_{p_ι}. Since $f_p = f^*_p \circ e$ for each p, we have

$$\begin{aligned} U = e^{-1} \circ e(U) \quad &= e^{-1}(e(X) \cap \cup_{\iota\in I} {f^*_{p_\iota}}^{-1}(U_{p_\iota})) \\ = X \cap e^{-1}(\cup_{\iota\in I} {f^*_{p_\iota}}^{-1}(U_{p_\iota})) \quad &= e^{-1}(\cup_{\iota\in I} {f^*_{p_\iota}}^{-1}(U_{p_\iota})) \\ = \cup_{\iota\in I} e^{-1}({f^*_{p_\iota}}^{-1}(U_{p_\iota})) \quad &= \cup_{\iota\in I} f^{-1}_{p_\iota}(U_{p_\iota}) \end{aligned}$$

Thus, X has the weak topology induced by $\mathcal{A}$.

Conversely, we assume that X has the weak topology induced by $\mathcal{A}$. First, we show that e is continuous. Let (x_n) be a sequence in X such that (x_n) converges to $x \in X$. Since $f_p = f^*_p \circ e$ for each p,

$$\lim f^*_p(e(x_n)) = \lim f_p(x_n) = f_p(x) = f^*_p(e(x))$$

for each p, and so $(f^*_p(e(x_n)))$ converges to $f^*_p(e(x))$ for each p. Since $X^*(\mathcal{A})$ has the weak topology induced by $\mathcal{A}^*$, $(e(x_n))$ converges to $e(x)$, and thus e is continuous. Next, to show that $e(X)$ is dense in $X^*(\mathcal{A})$, let $[(x_n)] \in X^*(\mathcal{A})$. Then, clearly, $(f^*_p(e(x_n)))$ converges to $f^*_p([(x_n)])$, and since $X^*(\mathcal{A})$ has the weak topology induced by $\mathcal{A}^*$, we have that $(e(x_n))$ converges to $[(x_n)]$. Thus $[(x_n)] \in \mathrm{Cl}_{X^*(\mathcal{A})}(e(X))$. Finally, we show that e is an open mapping from X onto $e(X)$. Since X has the weak topology induced by $\mathcal{A}$, we have that $e^{-1} : e(X) \longrightarrow X$ is continuous if and only if $f_p \circ e^{-1}$ is continuous for each p (See [3], p.56). Since $f_p \circ e^{-1} = f^*_p$ for each p, e^{-1} is continuous.

PROPOSITION 5. *Every $\mathcal{A}^*$-sequence in $X^*(\mathcal{A})$ has a cluster point in $X^*(\mathcal{A})$.*

Proof. Let (α_n) be an $\mathcal{A}^*$-sequence $X^*(\mathcal{A})$. Then, by definition of $\mathcal{A}^*$-sequence, $(f_p^*(\alpha_n))$ has a cluster point z_p in X_p for each $f_p^* \in \mathcal{A}^*$, and so there exists a subsequence $(\alpha_{p(i)})$ of (α_n) such that $(f_p^*(\alpha_{p(i)}))$ converges to z_p for each p. Since $\alpha_{p(i)} \in X^*(\mathcal{A})$ for each $p(i)$, there exists an element $(x_{p(i)}^m)$ of S such that $\alpha_{p(i)} = [(x_{p(i)}^m)]$ for each $p(i)$. It follows that $(f_p(x_{p(i)}^m))$ converges to $f_p^*(\alpha_{p(i)})$ for each p. Let $B_p = \{f_p(x_{p(i)}^m) \mid m, i \in N\}$. Then $f_p^*(\alpha_{p(i)}) \in \mathrm{Cl}_{X_p}(B_p)$ for each p, and hence we have $z_p \in \mathrm{Cl}_{X_p}(B_p)$. Thus there exists a sequence $(f_p(x_p^m))$ in B_p such that $(f_p(x_p^m))$ converges to z_p for each p. Let (l_r) be the sequence in X such that $l_1 = x_1^1, l_2 = x_2^1, \cdots, l_k = x_k^1, l_{k+1} = x_1^2, \cdots, l_{2k} = x_k^2, l_{2k+1} = x_1^3, \cdots$. Then, as the same technique in Lemma, there is a subsequence $(x_{p_0}^{n(j)})$ of $(x_{p_0}^n)$ such that $(f_{p_0}(x_{p_0}^{n(j)}))$ converges to z_{p_0}, for some p_0. Thus we have that

$$f_{p_0}^*([(x_{p_0}^{n(j)})]) = \lim f_p(x_{p_0}^{n(j)}) = z_{p_0} = \lim f_{p_0}^*(\alpha_{p_0(i)}).$$

Since X^* has the weak topology induced by $\mathcal{A}^*$, $(\alpha_{p_0(i)})$ converges to $[(x_{p_0}^{n(j)})]$, and therefore, (α_n) has a cluster point $[(x_{p_0}^{n(j)})]$.

PROPOSITION 6. *$X^*(\mathcal{A})$ is sequentially compact if and only if $Cl_{X_p}(f_p(X))$ is sequentially compact for each $f_p \in \mathcal{A}$.*

Proof. By Proposition 2 and 5, $X^*(\mathcal{A})$ is sequentially compact if and only if $\mathrm{Cl}_{X_p}(f_p^*(X^*(\mathcal{A})))$ is sequentially compact for each $f_p^* \in \mathcal{A}^*$. Since

$$f_p^*(X^*(\mathcal{A})) = f_p^*(\mathrm{Cl}_{X^*(\mathcal{A})}(e(X))) \subset \mathrm{Cl}_{X_p}(f_p^*(e(X))) = \mathrm{Cl}_{X_p}(f_p(X))$$

and

$$f_p(X) = f_p^*(e(X)) \subset f_p^*(X^*(\mathcal{A})),$$

we have $\mathrm{Cl}_{X_p}(f_p^*(X^*(\mathcal{A}))) = \mathrm{Cl}_{X_p}(f_p(X))$ for each p. Thus $X^*(\mathcal{A})$ is sequentially compact if and only if $\mathrm{Cl}_{X_p}(f_p(X))$ is sequentially compact for each $f_p \in \mathcal{A}$.

By Proposition 3, 4 and 6, we have our main result.

THEOREM. *$X^*(\mathcal{A})$ is a sequential Hausdorff compactification satisfying the first axiom of countability of a Hausdorff space X satisfying the first axiom of countability if and only if*

(1) *$Cl_{X_p}(f_p(X))$ is sequentially compact for each $f_p \in \mathcal{A}$, and*
(2) *X has the weak topology induced by $\mathcal{A}$.*

EXAMPLE. Let R be the real line with the usual topology and $X = [0,1)$. Define $f : X \longrightarrow R$ by $f(x) = x$ for all $x \in X$. Then, clearly, X has the weak topology induced by $\{f\}$ and $\mathrm{Cl}_R(f(X)) = [0,1]$ is sequentially compact. Since $S = \{(x_n) \mid (x_n)$ is a sequence in X with (x_n) converges in $R\}$ and $X^*(\{f\}) = \{[(x_n)] \mid (x_n)$ is a sequence in X with (x_n) converges in $X\} \cup \{[(x_n)] \mid (x_n)$ is a sequence in X with $\lim x_n = 1$ in $R\}$, hence we have that $X^*(\{f\})$ is homeomorphic to $[0,1]$ by the homeomorphism $h : X^*(f) \longrightarrow [0,1]$ defined by $h([(x_n)]) = \lim x_n$ in R. Therefore, $X^*(\{f\})$ is homeomorphic to the one-point compactification of X.

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Department of Mathematics
Pusan National University
Pusan 609–735, Korea