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CAUCHY DECOMPOSITION FORMULAS FOR SCHUR MODULES

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1. Introduction

The characteristic free representation theory of the general linear group is one of the powerful tools in the study of invariant theory, algebraic geometry, and commutative algebra. Recently the study of such representations became a popular theme. In this paper we study the representation-theoretic structures of the symmetric algebra and the exterior algebra over a commutative ring with unity 1. We shall first illustrate with some simple examples.

If F and G are two finitely generated free modules over a commutative ring R, then the symmetric algebra $S(F \otimes G)$ is naturally a $GL(F) \times GL(G)$-module. When R is a field of characteristic zero the symmetric algebra decomposes into a direct sum of all irreducible polynomial $GL(F) \times GL(G)$-modules:

$$S(F \otimes G) = \sum_{\lambda} L_\lambda F \otimes L_\lambda G.$$

This is equivalent to the identity on symmetric functions which is due to Cauchy. This is a special case of the notion of plethysm of Schur functions[8]. Notice that the above decomposition is not true over an arbitrary commutative ring because the modules $L_\lambda F \otimes L_\lambda G$ need not be irreducible. Over an arbitrary commutative ring such decomposition holds only up to a natural filtration (where "natural" means that all modules of this filtration have the structure of $GL(F) \times GL(G)$-modules).

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Another context, in which we will be mainly concerned in this paper, is that of realizing the plethysm formulas in the category of homogeneous polynomial representations of the general linear group. More precisely, are there natural decompositions of $S_k(\wedge^2 F)$ and $S_k(S_2 F)$?

D.A.Buchsbaum asked whether both the modules admits natural filtrations whose associated graded modules are isomorphic to direct sums of Schur modules. De Concini and Procesi essentially constructed a natural filtration for $S_k(S_2 F)$ in [4]. This affirms his question. However, we prove in this paper that the answer is negative in general.

Section 2 is a review of background material on Schur modules and Weyl modules, as well as Schur complexes.

In Section 3, we provide Pieri formulas in a form that is valid over any commutative ring.

In Section 4, we prove that the universal filtration for $S_k(\wedge^2 F)$ does not exist in the category of the homogeneous polynomial representations of the general linear group over *an arbitrary commutative ring* .

2. Preliminaries

Throughout this paper, unless otherwise specified, we adopt the definitions and notations of [1] and [2]. We also quote without proof several results contained therein.

Let N^∞ denote the set of all finite sequences of nonnegative integers, e.g., $\lambda = (\lambda_1, \lambda_2, \cdots)$, with $\lambda_i = 0$ for almost all indices i. If $\lambda \in N^\infty$, the *dual* or *conjugate* of λ is defined by $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots)$, where $\tilde{\lambda}_j$ = number of i's with $\lambda_i \geq j$. A *partition* is a finite sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ of nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. We say that the number of nonzero terms in a sequence $\lambda \in N^\infty$ is its *length*. The *weight* of a sequence λ in N^∞ is the sum of the terms of λ and is denoted by $|\lambda|$.

A *relative sequence* is a pair (λ, μ) of sequences in N^∞ such that $\mu \subseteq \lambda$ meaning that $\mu_i \leq \lambda_i$ for all i. We will use the notation λ/μ to represent relative sequences. If both λ and μ are partitions, then the relative sequence λ/μ will be called a *skew partition*. Observe that a sequence λ in N^∞ can be regarded as a relative sequence $\lambda/(0)$. We will often identify relative sequences with their diagrams or their shape matrices. A *diagram* of a relative sequence $\lambda/\mu = (\lambda_1, \lambda_2, \cdots, \lambda_r)/(\mu_1, \mu_2, \cdots, \mu_r)$

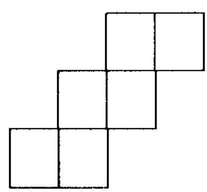
is a finite subset $\nabla_{\lambda/\mu} = \{(i,j) | 1 \le i \le r, \mu_i + 1 \le j \le \lambda_i\}$ of $N \times N$, but drawn as with matrices. The *shape matrix* of a relative sequence λ/μ is a $r \times \lambda_1$ matrix $A = (a_{ij})$ defined by the rule

$$a_{ij} = \begin{cases} 1 & \text{if } \mu_i + 1 \le j \le \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

As an example take $\lambda = (4,3,2)$ and $\mu = (2,1)$. Then the shape matrix of λ/μ is

$$\begin{bmatrix} 0\,0\,1\,1 \\ 0\,1\,1\,0 \\ 1\,1\,0\,0 \end{bmatrix},$$

and the diagram of λ/μ is



where each box represents an ordered pair (i,j). The weight of a relative sequence λ/μ is defined to be $|\lambda| - |\mu|$ and is denoted by $|\lambda/\mu|$.

To an $r \times t$ shape matrix $A = (a_{ij})$ we associate the row sequence $\alpha_A = (\alpha_1, \alpha_2, \cdots, \alpha_r)$ of A, where $\alpha_i = \sum_{j=1}^{t} a_{ij}$ for $i = 1, \cdots, r$, and the column sequence $\beta_A = (\beta_1, \beta_2, \cdots, \beta_r)$ of A, where $\beta_j = \sum_{i=1}^{r} a_{ij}$ for $j = 1, \cdots, t$. Clearly, $\beta_A = \alpha_{\tilde{A}}$, where $\tilde{A}$ is the transpose of A.

Let R be a commutative ring and let $\phi : G \to F$ be any homomorphism of finitely generated free R-modules. We let $\wedge\phi$ and $S\phi$ denote the exterior and symmetric algebras on the map ϕ. $\wedge\phi$ is the antisymmetric tensor product $\wedge F \hat{\otimes} DG$ of the Hopf algebras $\wedge F$ and DG, and $S\phi$ is the usual tensor product $SF \otimes \wedge G$ of the Hopf algebras SF and $\wedge G$. $\wedge\phi$ and $S\phi$ are nonnegatively graded Hopf algebras (with the multiplication $\mathcal{M}$ and the comultiplications or diagonalizations $\triangle$)

$$\wedge\phi = \sum \wedge^k \phi, \qquad S\phi = \sum S_k \phi$$

meaning that the Hopf algebra structure maps are homogeneous with respect to the above gradings. Moreover, $\wedge\phi$ and $S\phi$ can be naturally made into chain complexes in manner compatible with their Hopf algebra structures. For descriptions of the boundary maps and for details on the Hopf algebra structures of $\wedge\phi$ and $S\phi$, see [2, Chap.V].

If $\alpha = (a_1, \cdots, a_r)$ is any sequence of nonnegative integers, we define complexes $\wedge_\alpha\phi$ and $S_\alpha\phi$

$$\begin{aligned}\wedge_\alpha\phi &= \wedge^{a_1}\phi \otimes \cdots \otimes \wedge^{a_r}\phi\\ S_\alpha\phi &= S_{a_1}\phi \otimes \cdots \otimes S_{a_1}\phi\end{aligned}$$

as tensor products of complexes over R. If A is an $r \times t$ shape matrix, we set

$$\begin{aligned}\wedge_A\phi &= \wedge_{(\alpha_1,\cdots,\alpha_r)}\phi\\ S_A\phi &= S_{(\alpha_1,\cdots,\alpha_r)}\phi\end{aligned}$$

where $(\alpha_1, \cdots, \alpha_r)$ is the row sequence of A. We can then define a map of chain complexes, called the Schur map,

$$d_A(\phi) : \wedge_A\phi \longrightarrow S_{\tilde{A}}\phi$$

to be the composition

$$\begin{aligned}\wedge_A\phi = \wedge_{\alpha_A}\phi \rightarrow&(\wedge^{a_{11}}\phi \otimes \cdots \otimes \wedge^{a_{1t}}\phi) \otimes \cdots \otimes (\wedge^{a_{r1}}\phi \otimes \cdots \otimes \wedge^{a_{rt}}\phi)\\ \cong&(\wedge^{a_{11}}\phi \otimes \cdots \otimes \wedge^{a_{r1}}\phi) \otimes \cdots \otimes (\wedge^{a_{1t}}\phi \otimes \cdots \otimes \wedge^{a_{rt}}\phi)\\ \cong&(S_{a_{11}}\phi \otimes \cdots \otimes S_{a_{r1}}\phi) \otimes \cdots \otimes (S_{a_{1t}}\phi \otimes \cdots \otimes S_{a_{rt}}\phi)\\ \rightarrow&S_{\beta_1}\phi \otimes \cdots \otimes S_{\beta_t}\phi = S_{\tilde{A}}\phi,\end{aligned}$$

where the first map is diagonalization, the second is the isomorphism rearranging terms, the third is the isomorphism identifying $\wedge^{a_{ij}}\phi$ with $S_{a_{ij}}\phi$ (for $a_{ij} = 0$ or 1), and the last map is multiplication. It should be observed that the Schur map $d_A(\phi)$ does not depend on ϕ, for only G and F are used. Since each of the maps comprising $d_A(\phi)$ is a map of complexes, the Schur map $d_A(\phi)$ is a morphism of complexes. Hence, its image is a complex and we make the following definition.

DEFINITION 2.1. *The image of $d_A(\phi)$, denoted by $L_A\phi$, is called the Schur complex on ϕ associated to the shape matrix A. When A is the shape matrix of a skew partition λ/μ , we write $L_{\lambda/\mu}\phi$ instead of $L_A\phi$. It should be noted that the Schur complexes $L_A\phi$ are the complexes of $GL(G)\times GL(F)$-modules ("$GL(\phi)$-complex", for short) because $\mathcal{M}$ and $\triangle$ are morphisms of $GL(\phi)$-complexs ("$GL(\phi)$-morphism", for short).*

If we restrict our attention to the maps of the form $\phi : O \to F$, then we write $d_A(F)$ for $d_A(\phi)$ and recover the usual Schur module L_AF (in dimension zero) as the image of $d_A(F)$. Similarly, if the map is of the form $\phi : G \to O$ we write $d'_A(G)$ for $d_A(\phi)$ and obtain the Weyl module K_AG (in dimension $|A|$) as the image of $d'_A(G)$. When R is a field of characteristic zero, $L_\lambda F$ is an irreducible homogeneous polynomial $GL(F)$-module of degree $|\lambda|$ corresponding to the partition λ. Schur and Weyl module are isomorphic over the rationals, but far from being isomorphic over the integers.

PROPOSION 2.2 [2, THEOREM V.1.10]. *For any R, ϕ, and λ/μ, $L_{\lambda/\mu}\phi$ is a complex of universally free R-modules.*

DEFINITION 2.3. *When a, b, and k are positive integers with $k \leq b$, we have the $GL(\phi)$-morphisms $\wedge^{a+k}\phi \otimes \wedge^{b-k}\phi \overset{\triangle\otimes 1}{\to} \wedge^a\phi \otimes \wedge^k\phi \otimes \wedge^{b-k}\phi \overset{1\otimes\mathcal{M}}{\to} \wedge^a\phi \otimes \wedge^b\phi$. This composite map will be denoted by $\square_k(\phi)$ or $\square_k$. Similarly when $\alpha = (\alpha_a, \cdots, \alpha_r)$ is a sequence of positive integers we define a $GL(\phi)$-morphism as*

$$\sum_{i=1}^{r-1}\sum_{v=1}^{\alpha_{i+1}} \wedge^{\alpha_1}\phi \otimes \cdots \otimes \wedge^{\alpha_{i-1}}\phi \otimes \wedge^{\alpha_i+v}\phi \otimes \wedge^{\alpha_{i+1}-v}\phi \otimes \wedge^{\alpha_{i+2}}\phi \otimes \cdots \otimes \wedge^{\alpha_r}\phi$$

$$\downarrow \qquad \sum_{i=1}^{r-1}\sum_{v=1}^{\alpha_{i+1}} 1 \otimes \cdots \otimes 1 \otimes \square_v \otimes 1 \otimes \cdots \otimes 1$$

$$\wedge_\alpha \phi = \wedge^{\alpha_1}\phi \otimes \cdots \otimes \wedge^{\alpha_r}\phi$$

and denote it by $\square_\alpha(\phi)$ or $\square_\alpha$.

PROPOSITION 2.4 [2, THEOREM V.1.10]. *For any skew partion λ/μ, the following sequence of $GL(\phi)$-morphisms is exact :*

$$O \to Im(\square_{\lambda/\mu}(\phi)) \to \wedge_{\lambda/\mu}\phi \overset{d_{\lambda/\mu}(\phi)}{\to} L_{\lambda/\mu}\phi \to O.$$

Suppose now that $\phi : G \to F$ is the direct sum $\phi_1 \oplus \phi_2$ of two maps $\phi_i : G_i \to F_i$, $i = 1, 2,$. For any nonnegative integer p we have the direct sum decomposition

$$\wedge^p \phi = \sum_{a+b=p} \wedge^a \phi_1 \otimes \wedge^b \phi_2 \tag{1}$$

of chain complexes. If $P = (p_1, \cdots, p_r) \in N^\infty$, then (1) immediately yields a natural direct sum decomposition of the chain complexes as follows

$$\wedge_P \phi = \sum \wedge_\alpha \phi_1 \otimes \wedge_\beta \phi_2, \tag{2}$$

where the sum is taken over all sequences $\alpha = (\alpha_1, \cdots, \alpha_r)$, and $\beta = (\beta_1, \cdots, \beta_r)$ in N^∞ such that $\alpha_i + \beta_i = p_i$ for $i = 1, \cdots, r$. Fix integers a and b such that $a+b = |P|$, and define $\wedge_P(\phi_1, \phi_2; a, b)$ to be $\sum \wedge_\alpha \phi_1 \otimes \wedge_\beta \phi_2$ over all sequences α, β satisfying $\alpha + \beta = P$, $|\alpha| = a$, and $|\beta| = b$. It follows from (2) that there is a direct sum decomposition $\wedge_P \phi = \sum_{a+b=|P|} \wedge_P (\phi_1, \phi_2; a, b)$. The above discussion and definitions may be repeated with S in place of $\wedge$.

If A is a shape matrix, then we can apply the above discussion to the row and column sequences of A to obtain two natural direct sum decompositions

$$\wedge_A \phi = \sum_{a+b=|A|} \wedge_A(\phi_1, \phi_2; a, b)$$

$$S_{\tilde{A}} \phi = \sum_{a+b=|\tilde{A}|} S_{\tilde{A}}(\phi_1, \phi_2, ; a, b)$$

of chain complexes. It is immediate from homogeneity that the Schur map $d_A(\phi)$ decomposes into a direct sum $\sum_{a+b=|A|} d_A(\phi_1, \phi_2; a, b)$ where each $d_A(\phi_1, \phi_2; a, b)$ is a map of complexes $\wedge_A(\phi_1, \phi_2; a, b) \to S_{\tilde{A}}(\phi_1, \phi_2; a, b)$. We can then define $L_A(\phi_1, \phi_2; a, b)$ to be the image of the map $d_A(\phi_1, \phi_2; a, b)$ and obtain the natural direct sum decomposition

$$L_A\phi = \sum_{a+b=|A|} L_A(\phi_1, \phi_2; a, b) \tag{3}$$

of complexes.

DEFINITION 2.5. *Let $\phi = \phi_1 \oplus \phi_2$ as above and let A be the shape matrix of a skew partition λ/μ. If γ is any partition such that $\mu \subseteq \gamma \subseteq \lambda$, define $GL(\phi)$-subcomplexes M_γ and $\dot{M}_\gamma$ of $L_A\phi$ as $M_\gamma = \sum_{\gamma \subseteq \sigma} d_A(\phi)(\wedge_{\sigma/\mu}\phi_1 \otimes \wedge_{\lambda/\sigma}\phi_2)$, $\dot{M}_\gamma = \sum_{\gamma \subsetneq \sigma} d_A(\phi)(\wedge_{\sigma/\mu}\phi_1 \otimes \wedge_{\lambda/\sigma}\phi_2)$ where the sums are over the partitions σ satisfying $\mu \subseteq \sigma \subseteq \lambda$.*

It is easy to see that $\dot{M}_\gamma \subseteq M_\gamma$ and $M\mu = L_A\phi$, and $M_\delta \subseteq \dot{M}_\gamma$ if $\gamma \subsetneq \delta$. So we have a natural filtration of $L_A\phi$:

$$M_\lambda \subseteq \cdots \subseteq \dot{M}_\gamma \subseteq M_\gamma \subseteq \cdots \subseteq M_\mu = L_A\phi.$$

PROPOSITION 2.6 [2. THEOREM V.1.13]. *If γ is any partition such that $\mu \subseteq \gamma \subseteq \lambda$, there exists an isomorphism of chain $GL(\phi)$-complexes, $M_\gamma/\dot{M}_\gamma \cong L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2$.*

Hence, the complexes $\{M_\gamma | \mu \subseteq \gamma \subseteq \lambda\}$ give a natural filtration of the complex $L_{\lambda/\mu}\phi$ whose associated graded complex is isomorphic to $\sum_{\mu \subseteq \gamma \subseteq \lambda} L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2$.

Form the decomposition result (3) for a direct sum, this proposition can be reformulated for convenience as follows.

COROLLARY 2.7 [6, COROLLARY 2.5]. *If A is the shape matrix corresponding to a skew partition λ/μ, and integers a, b satisfy $a + b = |A|$, then the complex $L_A(\phi_1, \phi_2; a, b)$ admits a natural filtration $\{M_\gamma(L_A(\phi_1, \phi_2; a, b)) | \mu \subseteq \gamma \subseteq \lambda, |\gamma/\mu| = a$, and $|\lambda/\gamma| = b\}$ by complexes so that the associated graded complex is isomorphic to*

$\sum L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2$ *where the sum is taken over all partitions* γ *such that* $\mu \subseteq \gamma \subseteq \lambda, |\gamma/\mu| = a,$ *and* $|\lambda/\gamma| = b$.

To complete this section, we stress the fact that all the same results for Schur and Weyl modules will follow by specialzation in which $G = 0$ or $F = 0$.

3. Pieri Formulas

In this section, we provide Pieri formulas, which play an essential role in Section 4. Essentially, Pieri formulas for Schur complexes were constructured in [6]. Specializing these formulas into the formulas for Schur modules, we obtain Pieri formulas for Schur modules constructed in [1].

THEOREM 3.1. *Let* $\phi : G \to F$ *be any homomorphism of finitely generated free modules over an arbitrary commutative ring* R, *and let* λ *be any partition. Then*

(a) $L_{\lambda/(1^p)}\phi$ *admits a natural filtration whose associated graded complex is isomorphic to* $\sum\limits_{\mu} L_\mu\phi$ *where the sum is taken over all partitions* $\mu \subseteq \lambda$ *such that* $|\mu| = |\lambda| - p$ *and* $\lambda_i - 1 \leq \mu_i \leq \lambda_i$, *for all* i;

(b) $L_{\lambda/(p)}\phi$ *admits a natural filtration whose associated graded complex is isomorphic to* $\sum\limits_{\nu} L_\nu\phi$ *where the sum is taken over all partitions* $\nu \subseteq \lambda$ *such that* $|\nu| = |\lambda| - p$ *and* $\tilde{\lambda}_j - 1 \leq \tilde{\nu}_j \leq \tilde{\lambda}_j$ *for all* j.

Proof. **(a)** Let us take $\psi : O \to R$ to be the zero map. Using Corollary 2.7 we know that $L_\lambda(\psi, \phi; p, |\lambda| - p)$ has two natural filtrations which yield the following two decompositions: $L_\lambda(\psi, \phi; p, |\lambda| - p) \cong \sum\limits_{\substack{\gamma \subseteq \lambda \\ |\gamma| = p}} L_\gamma\psi \otimes L_{\lambda/\gamma}\phi$ up to filtration, $L_\lambda(\phi, \psi; |\lambda| - p, p) \cong \sum\limits_{\substack{\mu \subseteq \lambda \\ |\lambda/\mu| = p}} L_\mu\phi \otimes L_{\lambda/\mu}\psi$ up to filtration.

Since $\psi : O \to R$ is the zero map, $L_\gamma\psi = 0$ unless γ is the partition (1^p), and in this case $L_{(1^p)}\psi \cong (O \to R \to O)$. Therefore, we have $L_\lambda(\psi, \phi; p, |\lambda| - p) \cong L_{\lambda/(1^p)}\phi$. On the other hand, we have $L_{\lambda/\mu}\psi = 0$

unless λ/μ contains at most one box in each row, in which case $L_{\lambda/\mu}\psi \cong (O \to R \to O)$. Hence, we get exactly the statement (a).

(b) The formula (b) follows eaisly from the same line of reasoning as that of formula (a) with a zero map $\psi : R \to O$ (See the difference in the zero maps here and there).

If $\lambda = (\lambda_1, \cdots, \lambda_r)$ is a partition with $\lambda_r = 0$, then it follows from the definition of the Schur complexes that $L_\lambda\phi \otimes S_p\phi$ is isomorphic to $L_{\lambda'/(1^r)}$, where λ' is the partition $(\lambda_1 + 1, \lambda_2 + 1, \cdots, \lambda_r + 1, \underbrace{1, \cdots, 1}_{p})$. Also, $L_\lambda\phi \otimes \wedge^p\phi$ is isomorphic to $L_{\lambda''/(\lambda_1)}$, where $\lambda'' = (\lambda_1 + p, \lambda_1, \lambda_2, \cdots, \lambda_r)$. Hence we have Pieri formulas for Schur complexes.

COROLLARY 3.2. *Let $\phi : G \to F$ be any homomorphism of finitely generated free modules over an arbitrary commutative ring R, and let λ be any partition. Then*

(a) *$L_\lambda\phi \otimes S_p\phi$ admits a natural filtration whose associated graded complex is isomorphic to $\sum_\sigma L_\sigma\phi$ where the sum is taken over all partitions σ such that $|\sigma| = |\lambda| + p$ and $\lambda_i \le \sigma_i \le \lambda_i + 1$ for all i;*

(b) *$L_\lambda\phi \otimes \wedge^p\phi$ admits a natural filtration whose associated graded complex is isomorphic to $\sum_\gamma L_\gamma\phi$ where the sum is taken over all partitions γ such that $|\gamma| = |\lambda| + p$ and $\tilde{\lambda}_j \le \tilde{\gamma}_j \le \tilde{\lambda}_j + 1$ for all j.*

Specializing Pieri formulas in the formulas for Schur and Weyl modules which we need in the next section, we obtain

COROLLARY 3.3 [1,6]. *Let R be any commutative ring, F a free R-module of finite type, and λ any partition. Then*

(a) *$L_\lambda F \otimes S_p F$ admits a natural filtration whose associated graded module is isomorphic to $\sum_\sigma L_\sigma F$ where the sum is taken over all partitions σ such that $|\sigma| = |\lambda| + p$ and $\lambda_i \le \sigma_i \le \lambda_i + 1$ for all i;*

(b) *$L_\lambda F \otimes \wedge^p F$ admits a natural filtration whose associated graded module is isomorphic to $\sum_\gamma L_\gamma F$ where the sum is taken over all partitions γ such that $|\gamma| = |\lambda| + p$ and $\tilde{\lambda}_j \le \tilde{\gamma}_j \le \tilde{\lambda}_j + 1$ for all j.*

4. Plethysm formulas

If R contains the rationals Q, and k is any nonnegative integer, then we have the direct sum decompositions:

(1)
$$S_k(S_2F) = \sum_\lambda L_\lambda F.$$

where λ runs over all partitions of weight $2k$ such that each $\tilde{\lambda}_i$ is even;

(2)
$$S_k(\wedge^2 F) = \sum_\lambda L_\lambda F.$$

where λ runs over all partitions of weight $2k$ such that each λ_i is even;

(3)
$$\wedge^k(\wedge^2 F)) = \sum_\lambda L_{\tilde{\lambda}} F$$

where λ runs over all partitions of weight $2k$ such that the "Frobenius form" of λ is $(a_1, \cdots, a_r \mid a_1+1, \cdots, a_r+1)$ when the rank of λ is r.

Over an arbitrary commutative ring R, we may wonder whether they have (universally free) natural filtrations whose associated graded modules are isomorphic to the direct sums of the Schur modules described in (1), (2), and (3). In fact, we will prove in this section that $S_k(\wedge^2 F)$ and $\wedge^k(\wedge^2 F)$ do not admit such filtrations. But $S_k(S_2F)$ admits such a filtration.

PROPOSITION 4.1 [4,7]. *Let F be a finitely generated free module over an arbitrary commutative ring R, and let k be an arbitrary nonnegative integer. Then $S_k(S_2F)$ admits a natural filtration whose associated graded module is isomorphic to $\sum_\lambda L_\lambda F$, where λ runs over all partitions of weight $2k$ such that each $\tilde{\lambda}_i$ is even.*

The formulas as in Proposition 4.1 are called *plethysm formulas.* Now we will show that plethysm formulas for $\wedge^k(\wedge^2 F)$ and $S_k(\wedge^2 F)$ do not exist over *an arbitrary commutative ring* . To do this, we need the following proposition.

PROPOSITON 4.2. *Let F be a finitely generated free module over an arbitrary commutative ring R. Then $S_2(\wedge^2 F)$ admits a(universally free) natural filtration whose associated graded module is isomorphic to $\wedge^4 F \oplus L_{(2,2)}F$.*

Proof. To see this, we start with the natural Pfaffian embedding $\delta_2 : \wedge^4 F \to S_2(\wedge^2 F)$ which sends the elements $f_1 \wedge f_2 \wedge f_3 \wedge f_4$ in $\wedge^4 F$ to the Pfaffian $f_1 \wedge f_2 \cdot f_3 \wedge f_4 - f_1 \wedge f_3 \cdot f_2 \wedge f_4 + f_1 \wedge f_4 \cdot f_2 \wedge f_3$ in $S_2(\wedge^2 F)$, where $\cdot$ is a multiplication in $S_2(\wedge^2 F)$. Here we used the fact (proved in [4]) that the standard monomials form a free basis for $S(\wedge^2 F)$. Identifying $\wedge^4 F$ with its image in $S_2(\wedge^2 F)$, we get $\wedge^4 F$ with its image in $S_2(\wedge^2 F)$, we get $\wedge^4 F$ as the first piece of the filtration. Now we have to show that $S_2(\wedge^2 F)/ \wedge^4 F$ is isomorphic to $L_{(2,2)}F$. Consider a natural projection $\mathcal{M} : \wedge^2 F \otimes \wedge^2 F \longrightarrow S_2(\wedge^2 F)$ sending $f_1 \wedge f_2 \otimes f_3 \wedge f_4$ to $f_1 \wedge f_2 \cdot f_3 \wedge f_4$. Now we have the commutative diagram

$$\begin{array}{ccccc}
\wedge^4 F \oplus \wedge^3 F \otimes F & \xrightarrow{\square_{(2,2)}} & \wedge^2 F \otimes \wedge^2 F & \xrightarrow{d_{(2,2)}(F)} & L_{(2,2)}F \\
 & & \mathcal{M}\big\downarrow & \searrow^{\varphi_{(2,2)}} & \\
 & & S_2(\wedge^2 F) & \xrightarrow{\rho_{(2,2)}} & S_2(\wedge^2 F)/ \wedge^4 F
\end{array}$$

where $\rho_{(2,2)}$ is the projection and $\varphi_{(2,2)}$ is the composite map $\rho_{(2,2)} \cdot \mathcal{M}$. In order to construct the isomorphism $\psi : L_{(2,2)}F \to S_2(\wedge^2 F)/\wedge^4 F$, it is sufficient to show that $Ker(d_{(2,2)}(F)) = Ker(\varphi_{(2,2)})$, or equivalently $Im(\square_{(2,2)}) = Ker(\varphi_{(2,2)})$ by Proposition 2.4. But it is easy to see that $\mathcal{M} \cdot \square_{(2,2)}(f_1 \wedge f_2 \wedge f_3 \wedge f_4) = 2\delta_2(f_1 \wedge f_2 \wedge f_3 \wedge f_4) \in Im(\delta_2)$ and $\mathcal{M} \cdot \square_{(2,2)}(f_1 \wedge f_2 \wedge f_3 \otimes f_4) = \delta_2(f_1 \wedge f_2 \wedge f_3 \wedge f_4) \in Im(\delta_2)$. Thus we have an induced epimorphism $\psi : L_{(2,2)}F \to S_2(\wedge^2 F)/ \wedge^4 F$. Because $\wedge^4 F \oplus L_{(2,2)}F$ and $S_2(\wedge^2 F)$ are free modules of the same rank (by the universal freeness described in Porposition 2.2, the ranks of both the modules are independent of the ground ring R), the map ϕ must be an isomorphism. This completes the proof.

As a consequence of this proposition we have a short exact sequence

$$O \to \wedge^4 F \xrightarrow{\delta_2} S_2(\wedge^2 F) \xrightarrow{\psi^{-1} \cdot \rho_{(2,2)}} L_{(2,2)}F \to O.$$

Now we prove the main result.

THEOREM 4.2. *Let F be an n-dimensional vector space over a field R. Then*

(a) $\wedge^2(\wedge^2 F) = L_{(3,1)}F$ *if* $\mathrm{char} R \neq 2$;

(b) $\wedge^2(\wedge^2 F) = \wedge^4 F \oplus (K_{(2,1,1)}F/\wedge^4 F)$ *if* $\mathrm{char} R = 2$.

Proof. Consider the following commutative diagram, with exact rows and column:

$$\begin{array}{ccccccccc}
 & & & & O & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \wedge^2(\wedge^2 F) & & & & \\
 & & & & \downarrow & & & & \\
O & \longrightarrow & \wedge^4 F & \xrightarrow{h} & \wedge^4 F \oplus (\wedge^3 F \otimes F) & \xrightarrow{\square_{(2,2)}} & \wedge^2 F \otimes \wedge^2 F & \xrightarrow{d_{(2,2)}(F)} & L_{(2,2)}F & \longrightarrow & O \\
 & & \downarrow & & \downarrow g & & \downarrow & & \| & & \\
 & & O & \longrightarrow & \wedge^4 F & \longrightarrow & S_2(\wedge^2 F) & \longrightarrow & L_{(2,2)}F & \longrightarrow & O \\
 & & & & & & \downarrow & & & & \\
 & & & & & & O & & & &
\end{array}$$

where the bottom row is given by Proposition 4.2, and h sends an element x in $\wedge^4 F$ to the element $-2x+\Delta(x)$ in $\wedge^4 F \oplus (\wedge^3 F \otimes F)$,and g sends an element $x+y\otimes z$ in $\wedge^4 F \oplus (\wedge^3 F \otimes F)$ to the element $2x+y\wedge z$ in $\wedge^4 F$ (see also [9]).

(Here 2 and -2 stand for multiplication by 2 and -2, respectively.) Take now the kernel of the vertical columns: $\wedge^2(\wedge^2 F)$ is precisely the first homologgy module of

$$\mathbf{F} : \wedge^4 F \xrightarrow{h} \wedge^4 F \oplus (\wedge^3 F \otimes F) \xrightarrow{g} \wedge^4 F.$$

(a) Suppose that char$R \neq 2$. As the bottom row of

$$(4) \qquad \begin{array}{ccc} \wedge^4 F & \xrightarrow{\Delta} & \wedge^3 F \otimes F \\ {\scriptstyle -2}\downarrow & & \downarrow{\scriptstyle \mathcal{M}} \\ \wedge^4 F & \xrightarrow{2} & \wedge^4 F \end{array} \quad .$$

is an isomorphism, the first homology module of $\mathbf{F}$ is $(\wedge^3 F \otimes F)/Im(\Delta)$, i.e., $coker(\Delta) = L_{(3,1)}F$. Hence $\wedge^2(\wedge^2 F) = L_{(3,1)}F$.

(b) Suppose that char$R = 2$. As the bottom row of (4) is zero, $H_1(\mathbf{F})$ is $\wedge^4 F \oplus Ker(\wedge^3 F \otimes F \to \wedge^4 F)/\wedge^4 F$. Since

$$O \to D_4 F \to F \otimes D_3 F \to \wedge^2 F \otimes D_2 F \to \wedge^3 F \otimes F \to \wedge^4 F \to O$$

is exact (it is the dual of the Koszul complex), and since

$$Im(\wedge^2 F \otimes D_2 F \to \wedge^3 F \otimes F) = Im(d'_{(2,1,1)}(F)),$$

it follows $\wedge^2(\wedge^2 F) = \wedge^4 F \oplus K_{(2,1,1)}F/\wedge^4 F$.

As we have seen in the above proof, $\wedge^2(\wedge^2 F)$ has two different decompositions according to the characteristic of R. On the other hand, it follows from (3) (or by computing the characters [8]) that $\wedge^2(\wedge^2 F)$ is isomorphic to $L_{(3,1)}F$ over the field of characteristic zero. Therefore we have

COROLLARY 4.3. *There does not exist plethysm formulas of the form (3) for* $\wedge^k(\wedge^2_-)$ *which hold for any commutative ring* R *and any nonnegative integer* k.

Now we consider the module $S_k(\wedge^2 F)$. When R is a field of characteristic zero, $S_2(\wedge^2 F)$ is isomorphic to $\wedge^4 F \oplus L_{(2,2)}F$ by (2) (or by computing the characters [8]). But the Koszul complex

$$O \to \wedge^2(\wedge^2 F) \xrightarrow{\Delta} \wedge^2 F \otimes \wedge^2 F \xrightarrow{\mathcal{M}} S_2(\wedge^2 F) \to O$$

is universally free by Proposition 2.2. Furthermore, by Pieri formulas (Corollary 3.3. (b)), $\wedge^2 F \otimes \wedge^2 F$ admits a universally free natural filtration whose associated graded module is isomorphic to $\wedge^4 F \oplus L_{(3,1)}F \oplus L_{(2,2)}F$. Therefore, it is obvious that any universally free natural filtration of $S_2(\wedge^2 F)$ yields a universally free natural filtration of $\wedge^2(\wedge^2 F)$. But $\wedge^2(\wedge^2 F)$ does not have such a filtration by Corollary 4.3. Hence we have the result.

COROLLARY 4.4. *There does not exist plethysm formulas of the form (2) for* $S_k(\wedge^2_-)$ *which hold for any commutative ring* R *and any nonnegative integer* k.

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