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COMPOSITION WITH A HOMOGENEOUS POLYNOMIAL

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1. Introduction

Write B for the unit ball of $\mathbb{C}^n$ for a fixed integer $n \geq 2$ and let D denote the unit disc of $\mathbb{C}$. The Bloch space $\mathcal{B}$ is the space of functions f holomorphic on D such that

$$||f||_{\mathcal{B}} = \sup_{\lambda \in D}(1 - |\lambda|^2)|f'(\lambda)| < \infty.$$

For $f \in H^2$, the Hardy space on B, we say that $f \in BMOA$ if its radial limit function f^* is a function of bounded mean oscillations with respect to Lebesgue measure and nonisotropic balls that correspond to the Korányi approach regions. For details see [6].

In [1] Ahern proved that the monomial $\varphi(z) = n^{n/2} z_1 \cdots z_n$ which maps B onto D has the folllowing composition property:

$$f \circ \varphi \in BMOA \quad for \quad every \quad f \in \mathcal{B}.$$

Russo [8] applied the method of Ahern to obtain the same composition property for the homogeneous polynomial $\varphi : B \longrightarrow D$ defined by $\varphi(z) = z_1^2 + \cdots + z_n^2$. Ahern and Rudin [2] then noticed the fact that if φ is as above or a holomorphic monomial, then φ satisfies a sequence of equalities involving Cauchy integrals and utilized it in their new proof of the same composition property for such φ. Choe [4] used the method of Ahern-Rudin to prove the same composition property for functions belonging to a certain class of holomorphic homogeneous polynomials containing all the previous examples. In [4] it is pointed

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out that the methods of Ahern and Ahern-Rudin do not work for the following simple class of homogeneous polynomials :

$$\varphi(z) = a_1 z_1^d + \cdots + a_n z_n^d \qquad (d = 3,\ 4,\ \cdots). \tag{1}$$

where $|a_j| \le 1$ for $j = 1, \cdots, n$. In the present paper we use an entirely different method and prove that all the functions in (1) have the same composition property:

MAIN THEOREM. *Let φ be as in (1). Then $f \circ \varphi \in BMOA$ for every $f \in \mathcal{B}$.*

The proof of the main theorem yields another similar class of functions with the same composition property. See Remark at the end of the paper.

2. Proof of Main Theorem

We first introduce some notations. Let $S = \partial B$. The Lebesgue measure on S is denoted by σ. We let V denote the volume measure on B. For a holomorphic function f on B, we shall let $\nabla f = (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$ denote the complex gradient of f and let $\mathcal{R}f = \sum_{j=1}^n z_j(\partial f/\partial z_j)$ denote the radial derivative of f. The notation $< , >$ means the complex inner product on $\mathbb{C}^n$. For $z \in \mathbb{C}^n$, we let $|z| = < z, z >^{1/2}$.

A positive Borel measure μ on B is called a *Carleson measure* if

$$\mu(Q_\delta(\zeta)) = O(\delta^n)$$

where

$$Q_\delta(\zeta) = \{z \in B : |1- < z, \zeta > | < \delta\} \qquad (\delta > 0, \zeta \in S)$$

and the constant involved in the big "O" is independent of δ and ζ. The following characterization of the space $BMOA$ in terms of Carleson measures will play a key role in the proof of the main theorem.

THEOREM 1. *Suppose f is a function holomorphic on B. Then $f \in BMOA$ if and only if $\left(|\nabla f|^2 - |\mathcal{R} f|^2\right) dV$ is a Carleson measure.*

Proof. See [3].

Note. In [3] the above theorem is proved under the hypothesis $f \in H^2$. This hypothesis, however, can be easily removed as above, because the argument in [3] shows that a holomorphic function f on B is a member of H^2 if and only if $\left(|\nabla f|^2 - |\mathcal{R} f|^2\right) dV$ is a finite measure.

In the rest of the paper φ is fixed and denotes a function as in (1). Note that the main theorem is trivial if the sup-norm of φ is strictly less than 1. Therefore, by a unitary change of variables, we may assume without loss of generality that $a_1 = \cdots = a_m = 1$ and $|a_j| < 1$, $j = m+1, \cdots, n$ for some $1 \leq m \leq n$; thus

$$\varphi(z) = z_1^d + \cdots + z_m^d + a_{m+1} z_{m+1}^d + \cdots + a_n z_n^d. \tag{2}$$

The letter C will denote various constants, depending only on φ or n, which may change with each occurrence.

LEMMA 2. *There is a positive, radial, integrable, Borel function α on D such that*

$$\alpha(r) = C(1-r)^{n-2}[1 + o(1)] \qquad (r \nearrow 1)$$

with the following property: if h is a positive Borel function on D, then

$$\int_S h \circ \varphi \, d\sigma = \int_D h\alpha \, dA$$

where A denotes the area measure on D.

Proof. See [5, Theorem 3.1].

PROPOSITION 3. *There is a positive, radial, integrable, Borel function β on D such that*

$$\beta(r) = C(1-r)^{n-1}[1 + o(1)] \qquad (r \nearrow 1)$$

with the following property: if h is a positive Borel function on D, then

$$\int_B h\circ\varphi\,dV = \int_D h\beta\,dA.$$

Proof. Let α be the function introduced in Lemma 2 and fix a positive Borel function h on D. Integrating in polar coordinates and using Lemma 2, we obtain

$$\begin{aligned}\int_B h\circ\varphi\,dV &= C\int_0^1\int_S h(x^d\varphi)\,d\sigma\,x^{2n-1}dx\\ &= C\int_0^1\int_0^1\int_0^{2\pi} h(x^d ye^{i\theta})\,d\theta\,y\alpha(y)\,dy\,x^{2n-1}\,dx.\end{aligned}$$

Make successive changes of variables in the above integral: first $r = x^d$ and then $t = ry$ (r fixed). The result is

$$C\int_0^1\int_0^r\int_0^{2\pi} h(te^{i\theta})d\theta\,t\alpha(t/r)\,dt\,r^{(2n/d)-3}\,dr.$$

Accordingly, letting

$$\beta(\lambda) = C\int_{|\lambda|}^1 \alpha(\lambda/r)\,r^{(2n/d)-3}\,dr \qquad (\lambda\in D),$$

we obtain

$$\int_B h\circ\varphi\,dV = \int_D h\beta\,dA.$$

Clearly β is a positive, radial, integrable, Borel function on D. Note that

$$\alpha(t/r) = C(r-t)^{n-2}[1+o(1)] \qquad (t\nearrow 1)$$

uniformly in $r\ge t$ and hence

$$\begin{aligned}\beta(t) &= C[1+o(1)]\int_t^1 (r-t)^{n-2}\,dr\\ &= C(1-t)^{n-1}[1+o(1)]\end{aligned}$$

as desired. The proof is complete.

Let K denote the set of points $\zeta \in S$ for which $|\varphi(\zeta)| = 1$ and define

$$\rho(z) = \inf_{\eta \in K} |1- < z, \eta > | \qquad (z \in \bar{B}).$$

It is easily seen from (2) that $K = \cup_{j=1}^{m} K_j$ where K_j is the set of points $\zeta \in S$ such that $|\zeta_j| = 1$ and $\zeta_k = 0$ for $k \neq j$. It follows that

$$\rho(z) = \min_{1 \le j \le m} (1 - |z_j|).$$

For $\delta > 0$ and θ real, let

$$E_\delta(e^{i\theta}) = \{\lambda \in D : |1 - \lambda e^{-i\theta}| < \delta\}.$$

LEMMA 4. *Let $0 < \delta < 1/32$. Then we have the following.*

(*i*) *If $\zeta \in S$ and $\rho(\zeta) < 4\delta$, then $\varphi(Q_\delta(\zeta)) \subset E_{C\delta}(e^{i\theta})$ for some $e^{i\theta}$.*
(*ii*) *If $\zeta \in S$ and $4\delta \le \rho(\zeta) < 1/8$, then $1 - |\varphi| \ge C\delta$ on $Q_\delta(\zeta)$.*

Proof. Fix $0 < \delta < 1/32$, $\zeta \in S$, $z \in Q_\delta(\zeta)$, and let $\eta \in K$ be a point such that $\rho(\zeta) = |1- < \zeta, \eta > |$. First suppose $\rho(\zeta) < 4\delta$. Then the triangle inequality (see [7, Proposition 5.1.2.])
(3)

$$|1- < a, c > |^{1/2} \le |1- < a, b > |^{1/2} + |1- < b, c > |^{1/2} \quad (a,\ b,\ c \in \bar{B})$$

implies $|1- < z, \eta > | < 9\delta$. For simplicity assume $\eta = (e^{i\theta},\ 0, \cdots, 0)$ for some $e^{i\theta}$. We then have

$$\begin{aligned} |\varphi(\eta) - \varphi(z)| &\le |e^{id\theta} - z_1^d| + 1 - |z_1|^2 \\ &\le (d+2)|1 - z_1 e^{-i\theta}| \\ &= (d+2)|1- < z, \eta > | < 9(d+2)\delta, \end{aligned}$$

which shows $\varphi(Q_\delta(\zeta)) \subset E_{C\delta}(e^{id\theta})$. This proves (*i*).

Now suppose $4\delta \le \rho(\zeta) < 1/8$. Note that $\rho(z) < 1/2$ by (3). On the other hand,

$$\rho(z)^{1/2} \ge \rho(\zeta)^{1/2} - |1-<z,\zeta>|^{1/2} > \sqrt{\delta}$$

and thus $\rho(z) > \delta$. So $\delta < \rho(z) < 1/2$. For simplicity assume $\rho(z) = 1 - |z_1|$. Then $|\varphi(z)| \le |z_1|^d + 1 - |z_1|^2$ so that (recall $d \ge 3$)

$$1 - |\varphi(z)| \ge |z_1|^2(1 - |z_1|^{d-2}) \ge \frac{(1-|z_1|)}{4} > \frac{\delta}{4},$$

which shows (ii). The proof is complete.

We finally come to the proof of the main theorem.

Proof of Main Theorem. Let f be a Bloch function on D. Since $|\nabla\varphi| \le d$ on B and $\mathcal{R}\varphi = d\varphi$, we have

$$|\nabla(f\circ\varphi)|^2 - |\mathcal{R}(f\circ\varphi)|^2 = |f'(\varphi)|^2(|\nabla\varphi|^2 - |\mathcal{R}\varphi|^2) \le d^2\ ||f||_{\mathcal{B}}^2(1-|\varphi|)^{-1}.$$

Therefore, by Theorem 1, to prove $f\circ\varphi \in BMOA$, it is sufficient to show that $d\mu = (1-|\varphi|)^{-1}dV$ is a Carleson measure. Before going further, note that $V(Q_\delta(\zeta)) \le C\delta^{n+1}$ and

$$\int_{E_\delta(e^{i\theta})} (1-|\lambda|)^{-1}\ \beta\ dA \le C\delta^n. \tag{4}$$

where β is the function introduced in Proposition 3.

Since $(1-|\varphi|)^{-1}$ is integrable by Proposition 3, it is enough to consider δ sufficiently small. So assume $0 < \delta < 1/32$ and fix $\zeta \in S$. First consider the case $\rho(\zeta) < 4\delta$. Then, by Proposition 3, Lemma 4.(i) and (4), we see $\mu(Q_\delta(\zeta)) \le C\delta^n$. Next, consider the case $4\delta \le \rho(\zeta) < 1/8$. In this case, by Lemma 4.(ii), we have $\mu(Q_\delta(\zeta)) \le C\delta^{-1}V(Q_\delta(\zeta)) \le C\delta^n$. Finally, if $\rho(\zeta) \ge 1/8$, then $Q_\delta(\zeta) \subset \Gamma$ where $\Gamma = \{z \in \bar{B} : \rho(z) \ge 1/32\}$. Since $1 - |\varphi|$ has a positive minimum on Γ, we see that $\mu(Q_\delta(\zeta)) \le CV(Q_\delta(\zeta)) \le C\delta^{n+1} \le C\delta^n$. This completes the proof.

A close look at the proof of the main theorem gives another class of functions for which the composition property in question hold as follows.

REMARK. Suppose k and ℓ are positive integers such that $k+\ell=n$. Let h be a holomorphic homogeneous polynomial of degree $d \geq 3$ on $\mathbb{C}^\ell$ such that $h(B_\ell)=D$ and $|\nabla h| \leq d$ on B_ℓ. Let $a \in D$ and define

$$\psi(z) = z_1^d + \cdots + z_k^d + ah(z_{k+1}, \cdots, z_n) \qquad (z \in \mathbb{C}^n).$$

Then the analogues of Proposition 3 and Lemma 4 hold for ψ by exactly the same argument. Since $|\nabla h| \leq d$ on B_ℓ, we also have $|\nabla \psi| \leq d$ on B by the homogeneity of ∇h. We therefore conclude: *$f \circ \psi \in BMOA$ for every $f \in \mathcal{B}$.*

Note. After we completed the present paper, Wade Ramey and David Ullrich inform us that they have recently obtained a general result concerning the composition property in question by using an entirely different method.

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