Bull. Korean Math. Soc. **29** (1992), No. 1, pp. 73–80

TRANSVERSE HARMONIC FIELDS ON RIEMANNIAN MANIFOLDS

JIN SUK PAK AND HWAL-LAN YOO

We discuss transverse harmonic fields on compact foliated Riemannian manifolds, and give a necessary and sufficient condition for a transverse field to be a transverse harmonic one and the non-existence of transverse harmonic fields.

1. On a foliated Riemannian manifold, geometric transverse fields, that is, transverse Killing, affine, projective, conformal fields were discussed by Kamber and Tondeur ([3]), Molino ([5],[6]), Pak and Yorozu ([7]) and others. If the foliation is one by points, then transverse fields are usual fields on Riemannian manifolds. Thus it is natural to extend well known results concerning those fields on Riemannian manifolds to foliated cases.

On the other hand, the following theorem is well known ([1],[10]):

If the Ricci operator in a compact Riemannian manifold M is non-negative everywhere, then a harmonic vector field in M has a vanishing covariant derivative. If the Ricci operator in M is positive-definite, then a harmonic vector field other than zero does not exist in M.

In this paper we will extend the above result to foliated cases and prove the following theorems:

THEOREM A. *Let $(M, g_M, \mathtt{f})$ be a compact Riemannian manifold with harmonic foliation $\mathtt{f}$ and a bundle-like metric g_M with respect to $\mathtt{f}$. Let $s \in \overline{V}(\mathtt{f})$ be a transverse field of $\mathtt{f}$. Then s is a transverse harmonic field of $\mathtt{f}$ if and only if*

$$\Delta_D s + \rho_D(s) = 0.$$

Received March 7, 1991. Revised November 14, 1991.
This research was partially supported by TGRC-KOSEF.

THEOREM B. *Let (M, g_M, f) be as Theorem A. If the Ricci operator ρ_D is non-negative everywhere in M, then a transverse harmonic field is D-parallel. If ρ_D is non-negative everywhere and positive for at least one point of M, then a transverse harmonic field other than zero does not exist in M.*

We shall be in C^∞-category and deal only with connected and oriented manifolds without boundary. We use the following convention on the range of indices:

$$1 \le i, j \le p; \; p+1 \le a, b, c, d \le p+q$$

The Einstein summation convention will be used with respect to those systems of indices.

2. Let (M, g_M, f) be a $(p+q)$-dimensional Riemannian manifold with a foliation f of codimension q and a bundle-like g_M with respct to f ([8]). Let ∇ be the Levi-Civita connection with respect to g_M. Let TM be the tangent bundle of M and E the integrable subbundle of TM given by f. Then the *normal bundle* Q of f is defined by $Q = TM/E$. The metric g_M defines a splitting σ of the exact sequence

$$0 \to E \to TM \underset{\sigma}{\overset{\pi}{\rightleftarrows}} Q \to 0$$

with $\sigma(Q) = E^\perp$ (the orthogonal complement bundle of E in TM) ([2]). Then g_M induces a metric g_Q on Q;

$$g_Q(s,t) = g_M(\sigma(s), \sigma(t)), \quad s, t, \in \Gamma(Q), \tag{2.1}$$

where $\Gamma(*)$ denotes the set of all sections of $*$. In a flat chart $U(x^i, x^a)$ with respect to f ([8]), a local frame $\{X_i, X_a\} := \{\partial/\partial x^i, \partial/\partial x^a - A_a^j \partial/\partial x^j\}$ is called the *basic adapted frame* to f ([8],[9],[11]). Here A_a^j are functions on U with $g_M(X_i, X_a) = 0$. It is clear that $\{X_i\}$(resp. $\{X_a\}$) spans $\Gamma(E|_U)$(resp. $\Gamma(E^\perp|_U)$). We omit "$|_U$" for simplicity.
We set

$$\begin{aligned} g_{ji} = g_M(X_j, X_i), \quad g_{ba} = g_M(X_b, X_a), \\ (g^{ji}) = (g_{ji})^{-1}, \quad (g^{ba}) = (g_{ba})^{-1}. \end{aligned} \tag{2.2}$$

A connection D in Q is defined by

$$(2.3)\qquad \begin{aligned} D_X s &= \pi([X,Y]),\ X \in \Gamma(E) \\ &\qquad s \in \Gamma(Q) \text{ with } \pi(Y) = s \\ D_X s &= \pi(\nabla_X Y_s),\ X \in \Gamma(E^{\perp}) \\ &\qquad s \in \Gamma(Q) \text{ with } Y_s = \sigma(s) \end{aligned}$$

([2]). Then the connection D in Q is torsion-free and metrical with respect to g_Q ([2]). The curvature R_D of D is defined by

$$(2.4)\qquad R_D(X,Y)s = D_X D_Y s - D_Y D_X s - D_{[X,Y]} s$$

for any $X, Y \in \Gamma(TM)$ and $s \in \Gamma(Q)$. Since $i(X)R_D = 0$ for any $X \in \Gamma(E)$ ([2]), we can define the Ricci operator $\rho_D; \Gamma(Q) \to \Gamma(Q)$ of $\mathtt{f}$ by

$$(2.5)\qquad \rho_D(s) = g^{ba} R_D(\sigma(s), \pi(X_b))\pi(X_a)$$

([3]).
Let $V(\mathtt{f})$ be the space of all vector fields Y on M satisfying

$$(2.6)\qquad [Y, Z] \in \Gamma(E)$$

for any $Z \in \Gamma(E)$. An element of $V(\mathtt{f})$ is called an *infinitesimal automorphism* of $\mathtt{f}$ ([3],[6]). We set

$$(2.7)\qquad \overline{V}(\mathtt{f}) = \{s \in \Gamma(E) | s = \pi(Y), Y \in V(\mathtt{f})\}.$$

Then $s \in \overline{V}(\mathtt{f})$ satisfies

$$(2.8)\qquad D_X s = 0$$

for any $X \in \Gamma(E)$ ([3],[6]).

3. The transverse Lie derivative $\Theta(Y)$ with respect to $Y \in V(\mathtt{f})$ is defined by

$$(3.1)\qquad \Theta(Y)s = \pi([Y, Y_s])$$

for any $s \in \Gamma(Q)$ with $\pi(Y_s) = s$.

For $Y \in V(\mathbf{f})$, the operator $A_D(Y) : \Gamma(Q) \to \Gamma(Q)$ is defined by ([3])

$$A_D(Y)t = \Theta(Y)t - D_Y t. \tag{3.2}$$

Then we have

$$A_D(Y)t = -D_{Y_t}\pi(Y) \tag{3.3}$$

where $t = \pi(Y_t)$. This shows that

(i) $A_D(Y)$ depends only on $s = \pi(Y)$,
(ii) $A_D(Y)$ is a linear operator of $\Gamma(Q)$.

Thus we can use $A_D(s)$ instead of $A_D(Y)$ ([3]).

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. When M is compact, the inner product $\langle\langle , \rangle\rangle$ on $\Omega^r(M, Q)$ is defined by

$$\langle\langle t, u \rangle\rangle = \int_M g_Q(t \wedge *u) \tag{3.4}$$

([2]). Let $d_D : \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$ be the exterior differential operator and the operator $d_D{}^* : \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)$ is defined ([2]). If M is compact, then $d_D{}^*$ is the adjoint operator of d_D with respect to $\langle\langle , \rangle\rangle$ ([2]). The Laplacian Δ_D acting on $\Omega^r(M, Q)$ is defined by

$$\Delta_D = d_D d_D{}^* + d_D{}^* d_D. \tag{3.5}$$

An element of $\Gamma(Q)$ is regarded as an element of $\Omega^0(M, Q)$. The bundle map $\pi : TM \to Q$ is an element of $\Omega^1(M, Q)$. If $\Delta_D \pi = 0$, then the foliation $\mathbf{f}$ is said to be *harmonic*.

The Q-valued bilinear form α on M is defined by

$$\alpha(X, Y) = -(D_X \pi)(Y) \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$ ([2]). Since $\alpha(X, Y) = \pi(\nabla_X Y)$ for any $X, Y \in \Gamma(E)$, α is called the *second fundamental form* of $\mathbf{f}$ ([2]). The *tension field* τ of $\mathbf{f}$ is defined by

$$\tau = g^{ji}\alpha(X_j, X_i) \tag{3.7}$$

([2]). We remark that $\tau = d_D{}^*\pi \in \Gamma(Q)$. The foliation $\mathbf{f}$ is said to be *minimal* if $\tau = 0$. By the way, it is well known ([2]) that the harmonic foliation is equivalent to the minimal foliation.

Let $C^\infty(M)$ be the space of all functions on M. We define an operator $\mathrm{div}_D : \Gamma(Q) \to C^\infty(M)$ by

$$\mathrm{div}_D\, s = g^{ba} g_Q(D_{X_b} s, \pi(X_a)) \tag{3.8}$$

We call $\mathrm{div}_D\, s$ the *transverse divergence* of s with respect to D ([12]). Suppose that M is compact. Then

$$\int_M \mathrm{div}_D\, t\; dM = \langle\langle \tau, t \rangle\rangle$$

for any $t \in \Gamma(Q)$, and consequently, if $\mathbf{f}$ is harmonic, it follows that

$$\int_M \mathrm{div}_D\, t\; dM = 0 \tag{3.9}$$

for any $t \in \Gamma(Q)$ ([12]). Moreover, if M is compact, then it holds that

$$\langle\langle \Delta_D t, u \rangle\rangle = \langle\langle Dt, Du \rangle\rangle \text{ for any } t, u \in \overline{V}(\mathbf{f}),$$

where $\langle\langle Dt, Du \rangle\rangle = \int_M g^{ba} g_Q(D_{X_b} t, D_{X_a} u) dM$ ([12]).

We define the *transverse gradient* $\mathrm{grad}_D\, f$ of a function f with respect to D by

$$\mathrm{grad}_D\, f = g^{ba} X_b(f) \pi(X_a) \tag{3.10}$$

([7]). If M is compact, then we can obtain by the direct calculation ([7],[12])

$$g_Q(\mathrm{grad}_D\, \mathrm{div}_D\, t, t) = \sigma(t)(\mathrm{div}_D\, t), \tag{3.11}$$

$$\mathrm{div}_D((\mathrm{div}_D\, t)t) = \sigma(t)(\mathrm{div}_D\, t) + (\mathrm{div}_D\, t)^2, \tag{3.12}$$

$$\mathrm{div}_D(A_D(t)t) = -g^{ba} g_Q(D_{X_b} D_{\sigma(t)} t, \pi(X_a)), \tag{3.13}$$

$$\mathrm{Tr}(A_D(t) A_D(t)) = g^{ba} g_Q(D_{\sigma(\nabla_{X_b} \sigma(t))} t, \pi(X_a)) \tag{3.14}$$

for any $t \in \Gamma(Q)$, where Tr denotes the trace operator. By means of (3.9) and (3.12)-(3.14), we have

LEMMA 3.1. *([12]) Suppose that M is compact and $\mathbf{f}$ is minimal. Then it holds that*

$$\int_M \{\mathrm{Ric}_D(s) + \mathrm{Tr}(A_D(s)A_D(s) - (\mathrm{div}_D\, s)^2\} dM = 0$$

for any $s \in \overline{V}(\mathbf{f})$, where $\mathrm{Ric}_D(s) = g_Q(\rho_D(s), s)$.

4. Let ${}^tA_D(s)$ be the transpose of $A_D(s)$, that is, ${}^tA_D(s)$ satisfies the following equality:

$$g_Q(A_D(s)t, u) = g_Q(t, {}^tA_D(s)u)$$

for any $t, u \in \Gamma(Q)$. For $s \in \overline{V}(\mathbf{f})$, let $B_D(s) : \Gamma(Q) \to \Gamma(Q)$ be an operator defined by

$$B_D(s) = A_D(s) - {}^tA_D(S). \tag{4.1}$$

The operator $B_D(s)$ is skew-symmetric, that is,

$$g_Q(B_D(s)t, u) = -g_Q(t, B_D(s)u) \tag{4.2}$$

for any $t, u \in \Gamma(Q)$. Therefore, $\mathrm{Tr}(B_D(s)) = 0$.

On the other hand, by the direct calculation, we have

$$\mathrm{Tr}((B_D(s))^2) = 2\,\mathrm{Tr}(A_D(s)A_D(s)) - 2\,\mathrm{Tr}({}^tA_D(s)A_D(s)),$$

which together with Lemma 3.1 and the equality:

$$\int_M \mathrm{Tr}({}^tA_D(s)A_D(s))dM = \langle\langle Ds, Ds\rangle\rangle$$

yields

$$\begin{aligned}&\int_M \{\mathrm{Tr}({}^tB_D(S)B_D(s) + 2(\mathrm{div}_D\, s)^2\} dM \\ &= 2\int_M \{\mathrm{Ric}_D(s) + g_Q(Ds \wedge *Ds)\} dM\end{aligned} \tag{4.3}$$

because of (4.2)

DEFINITION. If $s \in \overline{V}(\mathtt{f})$ satisfies

$$B_D(s) = 0 \text{ and } \operatorname{div}_D s = 0,$$

then s is called a *transverse harmonic field* of $\mathtt{f}$.

Proof of Theorem A. Suppose that $\Delta_D s = -\rho_D(s)$. Since $\langle\langle Ds, Ds\rangle\rangle = \langle\langle \Delta_D s, s\rangle\rangle$, it follows that s is a transverse harmonic field. Conversely, if s is a transverse harmonic field, then we have

$$g_Q(D_{X_c}D_{X_b}s, \pi(X_a)) + g_Q(D_{X_b}s, D_{X_c}\pi(X_a)) - g_Q(D_{X_c}\pi(X_b), D_{X_a}s) \\ - g_Q(\pi(X_b), D_{X_c}D_{X_a}s) = 0,$$

from which, transvecting with g^{cb}, it follows that $\Delta_D s + \rho_D(s) = 0$ with the aid of $\operatorname{div}_D s = 0$. This completes the proof of Theorem A.

Next we will prove Theorem B.

Proof of Theorem B. If s is a transverse harmonic field, then we have

$$\int_M \{\mathrm{Ric}_D(s) + g_Q(Ds \wedge *Ds)\} dM = 0.$$

If the Ricci operator ρ_D of $\mathtt{f}$ is non-negative everywhere in M, then any transverse harmonic field s is D-parallel. Moreover, if ρ_D is positive at least one point of M, then any transverse harmonic fields s is zero, which completes the proof of Theorem B.

References

1. Bochner, S. *Vector fiedls and Ricci curvature,* Bull. Amer. Math. Soc. **52**(1946), 776–797.
2. Kamber, F. W. and Tondery, Ph. *Harmonic foliations,* Proc. NFS Conf. on Harmonic Maps, Tulane 1980. Lecture Notes in Math. 949, Springer-Verlag, Berlin-Heidelberg-New York 1982, 87–121.
3. Kamber, F. W. and Tondeur, Ph. *Infinitesimal automorphisms and second variation of the energy for harmonic foliations,* Tohoku Math. J. **34**(1982), 525–538.
4. Kobayashi, S. *Transformation groups in differential geometry,* Ergebnisse der Math. **70**. Springer-Verlag, Berlin-Heidelberg-New York 1972.
5. Molino, P. *Feuilletages Riemanniens sur les variétés compactes,* champs de Killing transverses. C. R. Acad. Sc. Paris **289**(1979), 421–423.

6. Molino, P. *Géométrie globale des feuilletages Riemanniens,* Proc. Kon. Ned. Akad. Al. **85**(1982), 45–76.
7. Pak, J. S. and Yorozu, S. *Transverse fields on foliated Riemannian manifolds,* J. Korean Math. Soc. **25**(1988), 83–92.
8. Reinhart, B. L. *Foliated manifolds with bundle-like metrics,* Ann. of Math. **69**(1959), 119–132.
9. Vaisman, I. *Cohomology and differential forms,* Marcel Dekker INC., New York 1973.
10. Yano, K. *Integral formulas in Riemannian geometry,* Marcel Dekker INC., New York 1970.
11. Yorozu, S. *Behavior of geodesics in foliated manifolds with bundle-like metrics,* J. Math. Soc. Japan **34**(1983), 251–272.
12. Yorozu, S. and Tanemura, T. *Green's theorem on a foliated Riemannian manifold and its applications,* Acta Math. Hung. **56**(1990).

Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea