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A CORRESPONDENCE BETWEEN HECKE RINGS $\mathcal{L}_0^n(q)$ AND $\mathcal{D}^n$

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0. Introduction and Notations

Hecke operators of degree n are closely related to polynomials in $\mathbf{C}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$. Many mathematicians like Satake [Sa], Shimura [Sh], Andrianov [A1], to name a few, did a great deal of work on this relation. One way of studying this relation is via the Hecke ring $\mathcal{D}_p^n$ associated to the Hecke pair (Λ^n, V^n), where $\Lambda^n = SL_n(\mathbf{Z})$ and $V^n = \{D \in M_n(\mathbf{Z}[p^{-1}]) \mid \det D = p^\delta, \ \delta \in \mathbf{Z}\}$.

The purpose of this article is to give a certain correspondence between Hecke operators and elements in $\mathcal{D}_p^n$, which is very interesting and useful in connection with the Hecke operators acting on Siegel modular forms.

Let $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ be the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. Let $\mathbf{F}_p$ be the field of p elements, where p is a prime.

Let $M_{m,n}(A)$ be the set of all $m \times n$ matrices over A, a commutative ring with 1, and let $M_n(A) = M_{n,n}(A)$. Let $GL_n(A)$ and $SL_n(A)$ be the group of invertible matrices in $M_n(A)$ and its subgroup consisting of matrices of determinant 1, respectively.

For $g \in M_m(A)$, $h \in M_{m,n}(A)$, let $g[h] = {}^t hgh$, where ${}^t h$ is the transpose of h. Let I_n and 0_n be the identity and the zero matrices, respectively. Let $\det g$ be the determinant of g. For $g \in M_{2n}(A)$, we let A_g, B_g, C_g, and D_g be the $n \times n$ block matrices in the upper left, upper right, lower left, and lower right corners of g, respectively, and write $g = (A_g, B_g; C_g, D_g)$. Let $\mathrm{diag}(N_1, N_2, \ldots, N_r)$ be the matrix with block matrices $N_1, N_2, \ldots, N_r$ on its main diagonal and zeroes outside.

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Let $\mathcal{N}_m$ be the set of all semi-positive definite (eigenvalues ≥ 0), semi-integral (diagonal entries and twice of nondiagonal entries are integers), symmetric $m \times m$ matrices, and $\mathcal{N}_m^+$ be its subset consisting of positive definite (eigenvalues > 0) matrices.

Let $G_n = GSp_n^+(\mathbf{R}) = \{g \in M_{2n}(\mathbf{R}) | J_n[g] = rJ_n,\ r > 0\}$ where $J_n = (0_n, I_n; -I_n, 0_n)$. r is a real number determined by g, which we often denote by $r(g)$. Let $\Gamma^n = Sp_n(\mathbf{Z}) = \{M \in M_{2n}(\mathbf{Z}) | J_n[M] = J_n\}$ and $L^n = L_p^n = \{g \in M_{2n}(\mathbf{Z}[\frac{1}{p}]) | J_n[g] = p^\delta J_n,\ \delta \in \mathbf{Z}\}$ where p is a prime. δ is an integer determined by g, which we often denote by $\delta(g)$.

For other general properties and terminologies, we refer the readers to [M], [A2].

1. Preliminaries

We introduce an abstract Hecke ring and some basic properties.

Let G be a multiplicative group and Γ be its subgroup. Let $\tilde{\Gamma}$ be the commensurator subgroup of Γ in G, i.e., $\tilde{\Gamma} = \{g \in G | g^{-1}\Gamma g \cap \Gamma$ is of finite index in both $g^{-1}\Gamma g$ and $\Gamma\}$. Let L be a semi-group containing Γ and contained in $\tilde{\Gamma}$. Let (Γ, L) be a Hecke pair, i.e., $\Gamma L = L\Gamma = L$. Let $V(\Gamma, L)$ be the vector space over $\mathbf{C}$ spanned by the formal left cosets (Γg), $g \in L$. For $X = \sum_{i=1}^{\mu} a_i(\Gamma g_i) \in V(\Gamma, L)$, $a_i \in \mathbf{C}$, and $g \in L$, we set $X \cdot g = \sum_{i=1}^{\mu} a_i(\Gamma g_i g) \in V(\Gamma, L)$. We define $\mathcal{L}(\Gamma, L) = \{X \in V(\Gamma, L) | X \cdot M = X$ for any $M \in \Gamma\}$. $\mathcal{L}(\Gamma, L)$ is in fact a ring under the multiplication defined by $X \cdot Y = \sum_{i,j} a_i b_j (\Gamma g_i h_j) \in \mathcal{L}(\Gamma, L)$ for any $X = \sum a_i(\Gamma g_i)$, $Y = \sum b_j(\Gamma h_j) \in \mathcal{L}(\Gamma, L)$. $\mathcal{L}(\Gamma, L)$ is called the Hecke ring of the Hecke pair (Γ, L).

We define a formal double coset $(\Gamma g \Gamma)$, $g \in L$, by $(\Gamma g \Gamma) = \sum_{i=1}^{\mu} (\Gamma g_i)$ when $\Gamma g \Gamma$ is a disjoint union of $\Gamma g_1, \ldots, \Gamma g_\mu$, $g_i \in L$. Since L is contained in $\tilde{\Gamma}$, $g^{-1}\Gamma g \cap \Gamma$ is of finite index in Γ and the index is exactly μ. If $M_1, \ldots, M_\mu$ are the left coset representatives of $g^{-1}\Gamma g \cap \Gamma$ in Γ, then $\Gamma g \Gamma$ is a disjoint union of $\Gamma g M_1, \ldots, \Gamma g M_\mu$, $gM_i \in L$, so that

$$(\Gamma g\Gamma) = \sum_{i=1}^{\mu}(\Gamma g M_i).$$

It is known [A1] that the formal left cosets $(\Gamma g\Gamma)$, $g \in L$, form a basis for $\mathcal{L}(\Gamma, L)$.

Let (Γ, L), (Γ', L') be Hecke pairs satisfying the conditions

$$\Gamma' \subset \Gamma,\ \Gamma L' = L, \text{ and } \Gamma \cap L'L'^{-1} \subset \Gamma'. \tag{1.1}$$

Then for $X = \sum a_i(\Gamma g_i) \in \mathcal{L}(\Gamma, L)$, g_i can be replaced by $g_i' \in L'$ because of the second condition so that X can be written in the form $X = \sum a_i(\Gamma g_i')$. We define a map $\varepsilon = \varepsilon(\mathcal{L}(\Gamma, L), \mathcal{L}(\Gamma', L')) : \mathcal{L}(\Gamma, L) \to \mathcal{L}(\Gamma', L')$ by

$$\varepsilon(X) = \sum a_i(\Gamma' g_i') \in \mathcal{L}(\Gamma', L'). \tag{1.2}$$

Then this map is an injective ring homomorphism. Moreover, it is an isomorphism if $[\Gamma : g'^{-1}\Gamma g' \cap \Gamma] = [\Gamma' : g'^{-1}\Gamma' g' \cap \Gamma']$ for every $g' \in L'$.

2. Hecke ring $\mathcal{L}_0^n(q, T)$

Let p be a prime and let n, q be positive integers such that $p \not| \, q$. We set

$$\Gamma_0^n(q) = \{M \in \Gamma^n \mid C_M \equiv 0 \pmod{q}\} \tag{2.1}$$

and

$$\Gamma_0^n = \{M \in \Gamma^n \mid C_M = 0\}. \tag{2.2}$$

We also set

$$L_0^n(q) = L_{0,p}^n(q) = \{g \in L^n \mid C_g \equiv 0 \pmod{q}\} \tag{2.3}$$

and

$$L_0^n = L_{0,p}^n = \{g \in L^n \mid C_g = 0\}. \tag{2.4}$$

It is well-known [A1] that (Γ^n, L^n), $(\Gamma_0^n(q), L_0^n(q))$, and (Γ_0^n, L_0^n) are all Hecke pairs and we denote the corresponding Hecke rings $\mathcal{L}(\Gamma^n, L^n)$, $\mathcal{L}(\Gamma_0^n(q), L_0^n(q))$, and $\mathcal{L}(\Gamma_0^n, L_0^n)$ by $\mathcal{L}^n = \mathcal{L}_p^n, \mathcal{L}_0^n(q) = \mathcal{L}_{0,p}^n(q)$, and $\mathcal{L}_0^n = \mathcal{L}_{0,p}^n$ respectively. Let $\Lambda^n = SL_n(\mathbf{Z})$ and let

$$V^n = V_p^n = \{D \in M_n(\mathbf{Z}[p^{-1}]) \mid \det D = p^\delta,\ \delta \in \mathbf{Z}\}. \tag{2.5}$$

Then (Λ^n, V^n) is also a Hecke pair and we denote the corresponding Hecke ring $\mathcal{L}(\Lambda^n, V^n)$ by $\mathcal{D}^n = \mathcal{D}_p^n$.

The pair $\{(\Gamma^n, L^n), (\Gamma_0^n(q), L_0^n(q))\}$ satisfies the conditions (1.1). So do the pairs $\{(\Gamma_0^n(q), L_0^n(q)), (\Gamma_0^n, L_0^n)\}$ and $\{(\Gamma^n, L^n), (\Gamma_0^n, L_0^n)\}$. Therefore, we have the following ring monomorphisms:

$$\alpha^n = \varepsilon(\mathcal{L}^n, \mathcal{L}_0^n(q)) : \mathcal{L}^n \to \mathcal{L}_0^n(q), \tag{2.6}$$

$$\beta^n = \varepsilon(\mathcal{L}_0^n(q), \mathcal{L}_0^n) : \mathcal{L}_0^n(q) \to \mathcal{L}_0^n. \tag{2.7}$$

and

$$\varepsilon^n = \varepsilon(\mathcal{L}^n, \mathcal{L}_0^n) : \mathcal{L}^n \to \mathcal{L}_0^n. \tag{2.8}$$

In fact, α^n is an isomorphism [A1].

It is obvious that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}^n & \xrightarrow{\varepsilon^n} & \mathcal{L}_0^n \\ \alpha^n \searrow & & \nearrow \beta^n \\ & \mathcal{L}_0^n(q) & \end{array} \tag{2.9}$$

We denote the image of $\mathcal{L}^n$ under ε^n by $\mathbf{L}_0^n$. From the above, it is clear that

$$\mathbf{L}_0^n = \varepsilon^n(\mathcal{L}^n) = \beta^n(\mathcal{L}_0^n(q)) \tag{2.10}$$

and that $\mathbf{L}_0^n$ is a subrings of $\mathcal{L}_0^n$. We now introduce even subrings of the above Hecke rings. We set

$$E^n = E_p^n = \{g \in L^n \mid \delta(g) \in 2\mathbf{Z}\}, \tag{2.11}$$

$$E_0^n(q) = E_{0,p}^n(q) = \{g \in L_0^n(q) \mid \delta(g) \in 2\mathbf{Z}\}, \tag{2.12}$$

and

$$(2.13) \qquad E_0^n = E_{0,p}^n = \{g \in L_0^n \mid \delta(g) \in 2\mathbf{Z}\}.$$

Obvionsly, $E_0^n(q) = E^n \cap L_0^n(q)$ and $E_0^n = E^n \cap L_0^n$. Again (Γ^n, E^n), $(\Gamma_0^n(q),\ E_0^n(q))$, and (Γ_0^n, E_0^n) are all Hecke pairs. The corresponding Hecke rings will be denoted by $\mathcal{E}^n$, $\mathcal{E}_0^n(q)$, and $\mathcal{E}_0^n$, and they are the even subrings of $\mathcal{L}^n$, $\mathcal{L}_0^n(q)$, and $\mathcal{L}_0^n$, respectively. We set

$$(2.14) \qquad \mathbf{E}_0^n = \varepsilon^n(\mathcal{E}^n) = \beta^n(\mathcal{E}_0^n(q)).$$

This is the even subring of $\mathbf{L}_0^n$.

Let

$$(2.15) \qquad K_s^n = \mathrm{diag}(I_{n-s}, pI_s, p^2 I_{n-s}, pI_s)$$

for $s = 0, 1, 2, \ldots, n$. Note that $\delta(K_s^n) = 2$. We denote the double coset $(\Gamma_0^n(q) K_s^n \Gamma_0^n(q))$ in $\mathcal{L}_0^n(q)$ by $T(K_s^n)$. Let $\mathcal{L}_0^n(q, T)$ be the subring of $\mathcal{L}_0^n(q)$ generated by $T(K_0^n), T(K_1^n), \cdots, T(K_{n-1}^n)$, and $T(K_n^n)^{\pm 1}$, i.e.,

$$(2.16) \qquad \mathcal{L}_0^n(q, T) = \mathbf{C}[T(K_0^n), T(K_1^n), \cdots, T(K_{n-1}^n), T(K_n^n)^{\pm 1}].$$

It is well-known [Z1] that $\mathcal{L}_0^n(q, T) = \mathcal{E}_0^n(q)$.

Finally, we set

$$(2.17) \qquad \mathbf{T}(K_s^n) = \beta^n(T(K_s^n))$$

for $s = 0, 1, \cdots, n$.

We obviously have

$$(2.18) \qquad \mathbf{E}_0^n = \mathbf{C}[\mathbf{T}(K_0^n), \mathbf{T}(K_1^n), \cdots, \mathbf{T}(K_{n-1}^n), \mathbf{T}(K_n^n)^{\pm 1}].$$

3. Ring Homomorphisms $\omega_n, \varphi_n, \psi_n$

In this section, we introduce ring homomorphisms

$$\omega_n = \omega_{n,p} : \mathcal{L}_0^n \to \mathcal{D}^n[t^{\pm 1}], \tag{3.1}$$

$$\varphi_n = \varphi_{n,p} = \mathcal{D}^n[t^{\pm 1}] \to \mathbf{C}_n[\underline{x}], \tag{3.2}$$

and

$$\psi_n = \psi_{n,p} : \mathcal{L}_0^n \to \mathbf{C}_n[\underline{x}], \tag{3.3}$$

which play crucial roles in what follows. Here

$$\mathbf{C}_n[\underline{x}] = \mathbf{C}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}]. \tag{3.4}$$

Let $X \in \mathcal{L}_0^n$. X can be written in the form $X = \sum a_i(\Gamma_0^n g_i)$, $g_i = \begin{pmatrix} p^{\delta_i} D_i^* & B_i \\ 0 & D_i \end{pmatrix} \in L_0^n$ where $\delta_i = \delta(g_i) \in \mathbf{Z}$ and $D_i \in V^n$ for each i. Here $D^* = ({}^tD)^{-1}$.

We define

$$\omega_n(X) = \sum a_i t^{\delta_i}(\Lambda^n D_i) \in \mathcal{D}^n[t^{\pm 1}]. \tag{3.5}$$

Clearly, δ_i and $(\Lambda^n D_i)$ are invariants of the left coset $(\Gamma_0^n g_i)$ for each i. Hence ω_n is a well-defined ring homomorphism : $\mathcal{L}_0^n \to \mathcal{D}^n[t^{\pm 1}]$. ω_n, in fact, is an epimorphism [Sa].

Let $Z = \sum a_i t^{\delta_i}(\Lambda^n D_i) \in \mathcal{D}^n[t^{\pm 1}]$. D_i can be chosen to be upper triangular in the form

$$D_i = \begin{pmatrix} p^{d_{i_1}} & * & \dots & * \\ 0 & p^{d_{i_2}} & & \vdots \\ \vdots & & \ddots & * \\ 0 & \dots & 0 & p^{d_{i_n}} \end{pmatrix} \tag{3.6}$$

for each i. We define

$$\varphi_n(Z) = \sum a_i x_0^{\delta_i} \Pi_{j=1}^n (x_j p^{-j})^{d_{i_j}} \in \mathbf{C}_n[\underline{x}]. \tag{3.7}$$

Since $d_{i_1}, \cdots, d_{i_n}$ are invariants of the left coset $(\Lambda^n D_i)$ for each i, φ_n is also a well-defined ring homomorphism : $\mathcal{D}^n[t^{\pm 1}] \to \mathbf{C}_n[\underline{x}]$.

We now set

$$\psi_n = \varphi_n \circ \omega_n : \mathcal{L}_0^n \to \mathbf{C}_n[\underline{x}]. \tag{3.8}$$

Let S_n be the symmetric group on $x_1, \cdots, x_n$ and let W_n be the group of automorphisms of $\mathbf{C}_n[\underline{x}]$ generated by S_n and $\sigma_1, \cdots, \sigma_n$, where σ_i is an automorphism of $\mathbf{C}_n[\underline{x}]$ defined by

$$\sigma_i : x_0 \mapsto x_0 x_i,\ x_i \mapsto x_i^{-1},\ x_j \mapsto x_j,\ \forall j \neq 0, i, \tag{3.9}$$

for each $i = 1, 2, \ldots, n$. Let $S_n[\underline{x}]$, $W_n[\underline{x}]$ be the subrings of $\mathbf{C}_n[\underline{x}]$ consisting of all the elements that are invariant under S_n, W_n, respectively.

Satake proved the followings [Sa]:

$$\varphi_n : \mathcal{D}^n[t^{\pm 1}] \simeq S_n[\underline{x}] \tag{3.10}$$

and

$$\psi_n : \mathbf{L}_0^n \simeq W_n[\underline{x}]. \tag{3.11}$$

The generators of $W_n[\underline{x}]$ are explicitely known and they are

$$\Delta_n(\underline{x})^{\pm 1} = (x_0^2 x_1 \cdots x_n)^{\pm 1}. \tag{3.12}$$

$$T_n(\underline{x}) = x_0 \sum_{i=0}^{n} s^i(x_1, \cdots, x_n) = x_0 \Pi_{j=1}^{n}(1 + x_j) \tag{3.13}$$

and

$$R_n^i(\underline{x}) = s^i(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}),\ 1 \leq i \leq n-1, \tag{3.14}$$

where $s^i(-)$ is the elementary symmetric polynomial of homogeneous degree i in the corresponding variables.

Let W_n^+ be the automorphism group of $\mathbf{C}_n[\underline{x}]$ generated by W_n and σ_0, where σ_0 is an automorphism of $\mathbf{C}_n[\underline{x}]$ defined by

$$\sigma_0 : x_0 \mapsto -x_0,\ x_i \mapsto x_i,\ \forall i \neq 0, \tag{3.15}$$

and let $W_n^+[\underline{x}]$ be the subring of $\mathbf{C}_n[\underline{x}]$ consisting of W_n^+-invariant elements. Then we have

$$\psi_n : \mathbf{E}_0^n \simeq W_n^+[\underline{x}]. \tag{3.16}$$

$W_n^+[\underline{x}]$ is generated by $\Delta_n(\underline{x})^{\pm 1}$, $T_n(\underline{x})^2$, and $R_n^i(\underline{x})$ for $i = 1, 2, \cdots, n-1$.

4. $l(r,i)$

Let $0 \le r \le i$ be integers. We set

$$l(r,i) = l_p(r,i) = |\{A = {}^tA \in M_i(\mathbf{F}_p) | r_p(A) = r\}|, \tag{4.1}$$

where $r_p(A)$ is the rank of A over $\mathbf{F}_p$. Obviously, $l(0,i) = 1$. We list the following proposition on $l(r,i)$ that can be deduced from results of Carlitz and Zhuravlev:

PROPOSITION 4.1. *Let*

$$\varphi_r = \varphi_r(p) = \begin{cases} \prod_{1\le a\le r}^{r}(p^a - 1) & \text{for } r \ge 1 \\ 1 & \text{for } r = 0, \end{cases} \tag{4.2}$$

$$\varphi_r^+ = \varphi_r^+(p) = \begin{cases} \prod_{\substack{1\le a\le r \\ a:\text{even}}}(p^a - 1) & \text{for } r \ge 2 \\ 1 & \text{for } r = 0,1, \end{cases} \tag{4.3}$$

and let

$$\varphi_r^- = \varphi_r^-(p) = \begin{cases} \prod_{\substack{1\le a\le r \\ a:\text{odd}}}(p^a - 1) & \text{for } r \ge 1, \\ 1 & \text{for } r = 0. \end{cases} \tag{4.4}$$

Then

$$l(r,r) = \frac{\varphi_r}{\varphi_r^+} p^{[\frac{r}{2}]([\frac{r}{2}]+1)} = \varphi_r^- p^{[\frac{r}{2}]([\frac{r}{2}]+1)} \tag{4.5}$$

and

$$l(r,i) = \frac{\varphi_i}{\varphi_{i-r}\varphi_r} l(r,r), \tag{4.6}$$

where [ ] *is the greatest integer function.*

Proof. See [C] and [Z2].

Let

$$D_{ij}^n = \operatorname{diag}(I_{n-i-j}, pI_i, p^2 I_j) \in V^n \tag{4.7}$$

(see (2.5)) and let

$$M_{ij}^n(A) = \begin{pmatrix} p^2 D_{ij}^* & X_{ij}(A) \\ 0 & D_{ij} \end{pmatrix} \in E_0^n \tag{4.8}$$

where $X_{ij}(A) = \mathrm{diag}(0_{n-i-j}, A, 0_j) \in M_n(\mathbf{Z})$ for $A = {}^tA \in M_i(\mathbf{Z})$. We fix a complete set R_{ij} of representatives of left cosets

$$(\Lambda^n \cap D_{ij}^n \Lambda^n (D_{ij}^n)^{-1}) \backslash \Lambda^n$$

for each $i, j \geq 0$ with $i + j \leq n$. Let

$$K_U^n = \begin{pmatrix} U^* & 0 \\ 0 & U \end{pmatrix} \in \Gamma_0^n \quad \text{for} \quad U \in GL_n(\mathbf{Z}) \tag{4.9}$$

and

$$P_B^n = \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} \in \Gamma_0^n \tag{4.10}$$

where $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & {}^tB_1 \\ 0 & B_1 & B_2 \end{pmatrix} \in M_n(\mathbf{Z})$ for $B_1 \in M_{ji}(\mathbf{Z})$ and $B_2 = {}^tB_2 \in M_j(\mathbf{Z})$. The following proposition is due to Zhuravlev:

PROPOSITION 4.2. *Let $M_{ij}^n(A, B, U) = M_{ij}^n(A) P_B^n K_U^n$. Then*

$$T(K_s^n) = \sum_{(*)} (\Gamma_0^n(q) M_{ij}^n(A, B, U)) \tag{4.11}$$

where the summation is over

(4.12)

$$(*) \begin{cases} i, j \geq 0 \text{ such that } i + j \leq n; \\ \text{for } A, A = {}^tA \in M_i(\mathbf{Z})/\mathrm{mod}\, p \text{ such that } r_p(A) = i - s; \\ \text{for } B, B_1 \in M_{ji}(\mathbf{Z})/\mathrm{mod}\, p \text{ and } B_2 = {}^tB_2 \in M_j(\mathbf{Z})/\mathrm{mod}\, p^2; \\ \text{for } U,\ U \in R_{ij}. \end{cases}$$

Proof. See [Z2].

We set

$$\Pi^n_{ij}(A) = (\Gamma^n_0 M^n_{ij}(A)\Gamma^n_0) \in \mathcal{E}^n_0. \tag{4.13}$$

For any symmetric $A_1, A_2 \in M_i(\mathbf{Z})$, one can easily veryfy that $\Pi_{ij}(A_1) = \Pi_{ij}(A_2)$ if and only if

$$A_1 \equiv A_2[U](\operatorname{mod} p) \quad \text{for some} \quad U \in GL_i(\mathbf{Z}). \tag{4.14}$$

(4.14) is an equivalent relation. We let $\{A\}$ be the equivalent class of A, which is called the p-class of A. Clearly $r_p(A)$ is an invariant of the p-class of A. We denote this by $r_p\{A\}$. We set

$$\Pi^{n,r}_{ij} = \sum_{\substack{\{A\} \\ r_p\{A\}=r}} \Pi^n_{ij}(A) \tag{4.15}$$

for each r, $0 \le r \le i$.

PROPOSITION 4.3.

$$\mathbf{T}(K^n_s) = \sum_{\substack{i\ge s, j\ge 0 \\ i+j\le n}} \Pi^{n,i-s}_{ij}. \tag{4.16}$$

Proof. From the definition (4.13) follows that for $A = {}^tA \in M_i(\mathbf{Z})$

$$\Pi^n_{ij}(A) = \sum_{\substack{B,U \\ C\in\{A\}/\operatorname{mod} p}} (\Gamma^n_0 M^n_{ij}(C, B, U)) \tag{4.17}$$

where B, U run over the matrices described in (4.12).

From Proposition 4.2 and (4.17) follows the proposition.

We set

$$Z^n_i = \sum_{0\le j\le n-i} p^{j(i+j+1)}(\Lambda^n D^n_{ij}\Lambda^n) \in \mathcal{D}^n. \tag{4.18}$$

THEOREM 4.4. *For* $s = 0, 1, 2, \cdots, n$.

$$\omega_n(\mathbf{T}(K_s^n)) = t^2 \sum_{s \le i \le n} l(i-s, i) Z_i^n. \tag{4.19}$$

Proof. From (4.15) and (4.17), we have

$$\Pi_{ij}^{n,r} = \sum_{\substack{A,B,U \\ r_p(A)=r}} (\Gamma_0^n M_{ij}^n(A, B, U)) \tag{4.20}$$

where $A = {}^tA \in M_i(\mathbf{Z})/\bmod p$ and B, U are matrices described in (4.12).

From the definition of ω_n and (4.20) follows

$$\omega_n(\Pi_{ij}^{n,r}) = t^2 \sum_{\substack{A,B,U \\ r_p(A)=r}} (\Lambda^n D_{ij}^n U). \tag{4.21}$$

The number of B's described in (4.12) is

$$p^{ij}(p^2)^{j(j+1)/2} = p^{j(i+j+1)} \text{ and } \sum_{U \in R_{ij}} (\Lambda^n D_{ij}^n U) = (\Lambda^n D_{ij}^n \Lambda^n).$$

So from the definition of $l(r, i)$, we obtain

$$\omega_n(\Pi_{ij}^{n,r}) = t^2 l(r, i) p^{j(i+j+1)} (\Lambda^n D_{ij}^n \Lambda^n).$$

The theorem follows immediately from Proposition 4.3.

This theorem in fact gives a complete description of the relation between $\mathbf{E}_0^n$ and $\mathcal{D}^n$ via ω_n because $\mathbf{E}_0^n$ is generated by $\mathbf{T}(K_0^n), \mathbf{T}(K_1^n), \cdots, \mathbf{T}(K_{n-1}^n)$, and $\mathbf{T}(K_n^n)^{\pm 1}$. See (2.18).

5. Main Theorem

The element $t^2 Z_0^n \in \mathcal{D}^n$ assumes an important role in connection with the Hecke operators acting on Siegel modular forms [Sc], [A3]. We now give an explicit formula for the Hecke operator in $\mathbf{E}_0^n$ that corresponds to $t^2 Z_0^n$ for each n via ω_n.

THEOREM 5.1. *Let*

$$\mathbf{T}^n = \mathbf{T}_p^n = \sum_{s=0}^{[\frac{n-1}{2}]} \frac{(-1)^s}{p^{s(s+1)}} \left(l(2s,2s)\cdot\mathbf{T}(K_{2s}^n) - l(2s+1,2s+1)\cdot\mathbf{T}(K_{2s+1}^n)\right) + \varepsilon(n)(-1)^{\frac{n}{2}} p^{-\frac{n}{2}(\frac{n}{2}+1)} l(n,n)\cdot\mathbf{T}(K_n^n) \in \mathbf{E}_0^n, \tag{5.1}$$

where

$$\varepsilon(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Then $\omega_n(\mathbf{T}^n) = t^2 Z_0^n \in \mathcal{D}^n[t^{\pm 1}]$.

Proof. From Theorem 4.4, we have $\omega_n(\mathbf{T}^n) = t^2(c_0 Z_0^n + c_1 Z_1^n + \cdots + c_n Z_n^n)$, where

$$c_i = \sum_{s=0}^{[\frac{i-1}{2}]} (-1)^s p^{-s(s+1)} \Big(l(2s,2s) l(i-2s,i) - l(2s+1,2s+1) l(i-(2s+1),i) \Big) + \varepsilon(i)(-1)^{\frac{i}{2}} p^{-\frac{i}{2}(\frac{i}{2}+1)} l(i,i) l(0,i)$$

for $i = 1, 2, \cdots, n$. Note that $c_0 = 1$. It suffices to show that $c_i = 0$ for all $i = 1, 2, \cdots, n$.

Let i be odd. Then the last term of c_i is 0. From Proposition 4.1 follows that for each $s = 0, \cdots, [\frac{i-1}{2}]$,

$$\begin{aligned} &l(2s,2s)l(i-2s,i) - l(2s+1,2s+1)l(i-(2s+1),i) \\ &\quad = l(2s,2s)l(i-2s-1,i)\Big((p^{i-(i-2s-1)}-1) - (p^{2s+1}-1)\Big) = 0. \end{aligned}$$

Hence $c_i = 0$ for all odd $i \geq 1$.

Let i be even. From a direct computation using Proposition 4.1, we get

$$c_i = \sum_{s=0}^{\frac{i}{2}} (-1)^s p^{-s(s-1)-i} l(2s,2s) l(i-2s,i).$$

Since

$$l(i-2s,i) = \frac{\varphi_i}{\varphi_{i-2s}\varphi_{2s}} l(i-2s,i-2s) = \frac{\varphi_i}{\varphi_{2s}\varphi_{i-2s}^+} p^{(\frac{i}{2}-s)(\frac{i}{2}-s+1)},$$

we have

$$c_i = p^{\frac{i}{2}(\frac{i}{2}-1)}\varphi_i^- \left(\sum_{s=0}^{\frac{i}{2}} \frac{\varphi_i^+}{\varphi_{i-2s}^+} \frac{l(2s,2s)}{\varphi_{2s}}(-p^{-i})^s\right). \tag{5.2}$$

We now prove the following identity by induction on j:

$$\sum_{s=0}^{j} \frac{\varphi_{2j}^+}{\varphi_{2j-2s}^+} \frac{l(2s,2s)}{\varphi_{2s}} z^s = \Pi_{s=1}^{j}(1+p^{2s}z) \tag{5.3}$$

For $j=1$, both sides of (5.3) equal to $1+p^2z$. Assuming (5.3) for $j-1$, we have

(5.4)

$$\begin{aligned}\Pi_{s=1}^{j}(1+p^{2s}z) &= (\Pi_{s=1}^{j-1}(1+p^{2s}z))(1+p^{2j}z)\\ &= \left(\sum_{s=0}^{j-1} \frac{\varphi_{2j-2}^+}{\varphi_{2j-2-2s}^+} \frac{l(2s,2s)}{\varphi_{2s}} z^s\right)(1+p^{2j}z).\end{aligned}$$

We now compare the coefficients of (5.3) and (5.4). For $s=0$, both coefficients are 1. For $s=j$,

$$\begin{aligned}&\left(\frac{\varphi_{2j-2}^+}{\varphi_0^+} \frac{l(2j-2,2j-s)}{\varphi_{2j-2}} p^{2i}\right)\left(\frac{\varphi_{2j}^+}{\varphi_0^+} \frac{l(2j,2j)}{\varphi_{2j}}\right)^{-1}\\ &= \frac{1}{p^{2j}-1} \frac{(p^{2j-1}-1)(p^{2j}-1)}{(p^{2j-1}-1)p^{2j}} p^{2j} = 1.\end{aligned}$$

For $1 \leq s \leq j-1$,

$$\begin{aligned}
&\left(\frac{\varphi^+_{2j-2}}{\varphi^+_{2j-2-2s}}\frac{l(2s,2s)}{\varphi_{2s}} + \frac{\varphi^+_{2j-2}}{\varphi^+_{2j-2s}}\frac{l(2s-2,2s-2)}{\varphi_{2s-2}}p^{2j}\right)\left(\frac{\varphi^+_{2j}}{\varphi^+_{2j-2s}}\frac{l(2s,2s)}{\varphi_{2s}}\right)^{-1} \\
&= \frac{\varphi^+_{2j-2}}{\varphi^+_{2j}}\cdot\frac{\varphi^+_{2j-2s}}{\varphi^+_{2j-2-2s}} + \frac{\varphi^+_{2j-2}}{\varphi^+_{2j}}\cdot\frac{\varphi_{2s}l(2s-2,2s-2)}{\varphi_{2s-2}l(2s,2s)}p^{2j} \\
&= \frac{p^{2j-2s}-1}{p^{2j}-1} + \frac{1}{p^{2j}-1}\frac{(p^{2s}-1)(p^{2s-1}-1)}{(p^{2s-1}-1)p^{2s}}p^{2j} \\
&= \frac{p^{2j-2s}-1+(p^{2s}-1)p^{2j-2s}}{p^{2j-1}} = 1
\end{aligned}$$

So we can conclude that the identity (5.3) holds. Substituting $j = \frac{i}{2}$ and $z = -p^{-i}$ in (5.3), we have $c_i = p^{\frac{i}{2}(\frac{i}{2}-1)}\varphi_i^- \Pi_{s=1}^{\frac{i}{2}}(1+p^{2s}(-p^{-i})) = 0$. Hence $c_i = 0$ for all even $i \geq 2$. This completes the proof.

Note that this correspondence between $\mathbf{T}^n \in \mathbf{E}_0^n$ and $t^2 Z_0^n \in \mathcal{D}^n$ can be extended to that between $T^n \in \mathcal{E}_0^n(q) = \mathcal{L}_0^n(q,T)$ and $t^2 Z_0^n \in \mathcal{D}^n$, where

$$\begin{aligned}
T^n = T_p^n = \sum_{s=0}^{[\frac{n-1}{2}]} (-1)^s p^{-s(s+1)}\Big(l(2s,2s)\cdot T(K_{2s}^n) \\
- l(2s+1,2s+1)\cdot T(K_{2s+1}^n)\Big) \\
+ \varepsilon(n)(-1)^{\frac{n}{2}}p^{-\frac{n}{2}(\frac{n}{2}+1)}l(n,n)\cdot T(K_n^n) \in \mathcal{L}_0^n(q,T).
\end{aligned} \tag{5.5}$$

Note that we have for $n = 1$

$$T^1 = T(K_0^1) - (p-1)T(K_1^1) \in \mathcal{L}_0^1(q,T). \tag{5.6}$$

This is a little bit different from the classical Hecke operator

$$T^1(p^2) = T(K_0^1) + T(K_1^1) \in \mathcal{L}_0^1(q,T). \tag{5.7}$$

As is well-known,

(5.8)

$$T^1(p^2) = \left(\Gamma_0^n(q)\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}\Gamma_0^n(q)\right) + \left(\Gamma_0^n(q)\begin{pmatrix}p & 0\\ 0 & p\end{pmatrix}\Gamma_0^n(q)\right)$$
$$= \sum_{\substack{ad=p^2\\ 0\le b<d}} \left(\Gamma_0^n(q)\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\right).$$

One can easily check that $\psi_1\circ\beta^1(T^1) = x_0^2(1+x_1^2)$ and $\psi_1\circ\beta^1(T^1(p^2)) = x_0^2(1+x_1+x_1^2)$. This difference appears in the eigenvalues for a simultaneous eigenform in the spaces $\mathcal{M}_k^1(q,\chi)$ of the classical modular forms of weight k, level q, with respect to a Dirichlet character χ modulo q. Indeed, if we let $f \in \mathcal{M}_k^1(q,\chi)$ be a simultaneous eigenform under the Hecke operators in $\mathcal{L}_0^n(q,T)$, then the eigenvalue of f for T^1 is $1 + (p^{k-1}\chi(p))^2$ while that for $T^1(p^2)$ is $1 + p^{k-1}\chi(p) + (p^{k-1}\chi(p))^2$. See [Se].

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