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UNIQUENESS FOR THE MARTINGALE PROBLEM WITH DISCONTINUOUS COEFFICIENT

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1. Introduction

Let L be the operator defined on $f \in C^2(R^d)$ by

$$Lf(x) = \sum_{i,j=1}^{d} \frac{1}{2} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

where a_{ij} are bounded and measurable and the matrix a is strictly elliptic. Then saying P_x is a solution of martingale problem with respect to L starting from x means that P_x is a measure on $C([0,\infty), R^d)$ such that

$$P_x(X_0 = x) = 1 \tag{1}$$

and

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds \tag{2}$$

is a P_x local martingale for all $f \in C^2(R^d)$. We denote the martingale problem with respect to L starting from x by $(MP)_x - L$. When a is continuous, we have existence and uniqueness of the solution of $(MP)_x - L$. When a is discontinuous, existence is known to hold while uniqueness remains open for $d \geq 3$.

Supposed the a_{ij} are bounded, strictly elliptic and continuous on $R^d \setminus \{0\}$ and define $a^n_{ij}(x) = a_{ij}(2^n x)$. Also suppose $a^n \longrightarrow a^\infty$ in measure

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on $B(0, R)$ for all $R < \infty$ where the a_{ij}^{∞} are continuous on $R^d \setminus \{0\}$, strictly ellipic and radially homogeneous. Then we prove uniqueness of martingale problem with respect to L in Section 3. And we get a result about uniform convergence of solution of Dirichlet problem in the unit ball.

In Section 2, we give some preliminaries and, in Section 3, we prove our main results.

2. Preliminaries

Suppose $a : R^d \longrightarrow R^{d \cdot d}$ is measurable, bounded and strictly elliptic: there is $\lambda > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x) y_i y_j \geq \lambda \sum_{i=1}^{d} y_i^2 \quad \text{for all } y_1, \cdots, y_d.$$

Let

$$(3) \qquad Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

Let $\Omega = C([0, \infty), R^d)$ and let $X_t(\omega) = \omega(t)$. We say a probability measure P satisfies the martingale problem for L_a starting from $x \in R^d$ if P is a measure satisfying the condition (1) and (2) replacing L with L_a. Recall we denote the martingale problem with respect to L_a by $(MP)_x - L_a$.

Saying P is unique means that any two solutions to the martingale problem for L_a agree on $\mathcal{F}_\infty = \sigma(X_t, t \in [0, \infty))$.

Let $S = \{x \in R^d : |x| = 1\}, B(x, R) = \{y \in R^d : |y - x| \leq R\}$ and $\tau_r = \inf\{t : |X_t| \geq r\}$. Our main result, proved in Section 3, is Theorem 2.1.

THEOREM 2.1. *Suppose the A_{ij} are bounded, strictly elliptrc, continuous on $R^d \setminus \{0\}$ and define $a_{ij}^n(x) = a_{ij}(2^n x)$. Suppose $a^n \longrightarrow a^\infty$ in measure on $B(0, R)$ for all $R < \infty$ where the a_{ij}^∞ are continuous on $R^d \setminus \{0\}$, strictly elliptic and radially homogeneous:*

$$a_{ij}^\infty(rx) = a_{ij}^\infty(x) \text{ for } r > 0.$$

Then P_x, the solution of the martingale problem with respect to L_a starting from x, is unique for all $x \in R^d$.

3. Uniqueness

Let L_{a^n} and L_{a^∞} be the operators defined by (3) with respect to a^n_{ij} and a^∞_{ij}. Then existence and uniqueness of the martingale problem with respect to L_{a^∞} is known (See [BP], [SV]). Also we have existence of the martingale problem for L_{a^n} and L_{a^∞} (see Ch.6 of [SV]). Let P^n_x be a solution for $(MP)_x - L_{a^n}$ and let P^∞_x be the unique solution for $(MP)_x - L_{a^\infty}$. Then $P^n_x \longrightarrow P^\infty_x$ weakly by Exercise 7.3.2 of [SV], since P^∞_x is unique and the inequality in (3.1) of Exercise 7.3.2 of [SV] holds for P^n_x and P^∞_x by Krylov [Kr]: for all $n \geq 1$ and $n = \infty$,

$$|E^{P^n_x} \int_0^T f(X_t)dt| \leq C\|f\|_{L^p(R^d)}$$

for all $p > d, T > 0, R > 0$ and $f \in C_0(R^d)$ with $\text{supp}(f) \subset B(0,R)$, where C depends only on the ellipticity and bounds on a_{ij}, T and R.

To show the uniqueness of $(MP)_x - L_a$, consider Q_n and Q_∞ defined on $C(S)$ by

$$Q_n(x,dy) = P^n_x(X_{\tau_2}/2 \in dy; \tau_2 < \tau_0)$$

and

$$Q_\infty(x,dy) = P^\infty_x(X_{\tau_2}/2 \in dy; \tau_2 < \tau_0).$$

for $x, y \in S$. Since X_t is continuous P^∞_x a.s., $\tau_\varepsilon \uparrow \tau_0$ as $\varepsilon \downarrow 0$. Therefore τ_0 is P^∞_x-continuous ([Ku], p.13). Then provided τ_2 is a P^∞_x-continuous functional on $C(R^d), Q_n f(x) \longrightarrow Q_\infty f(x)$ on S, since $P^n_x \longrightarrow P^\infty_x$ weakly. And to show that τ_2 is P^∞_x-continuous, it suffices to show that $\tau_{2+\varepsilon} \longrightarrow \tau_2$ a.s. P^∞_x as $\varepsilon \longrightarrow 0$.

LEMMA 3.1. $\tau_{2+\varepsilon} \longrightarrow \tau_2$ a.s. P^∞_x as $\varepsilon \longrightarrow 0$.

Proof. We know under P^∞_x, X_t solves

$$X_{t\wedge\tau_0} - x = \int_0^{t\wedge\tau_0} \sigma(X_s)dB_s$$

for some σ where $(\sigma\sigma^*)_{ij} = a_{ij}^{\infty}$ and B_t is a d-dimensional Brownian motion. X_t is a martingale and $< X_t^i, X_t^j >= \int_0^t a_{ij}^{\infty}(X_s)ds$. Let $R_t = |X_t|$. Then by Ito's formula, for $t < \tau_0$,

$$dR_t = |X_t|^{-1} X_t^* \sigma(X_t) dB_t + (2R_t)^{-1}(\text{ trace } \sigma\sigma^*(X_t) \\ - |X_t|^{-2} X_t^* \sigma\sigma^*(X_t) X_t) dt.$$

Let $A_t = \int_0^t |X_s|^{-2} X_s^* \sigma\sigma^*(X_s) X_s ds$. Then since $a_{ij}^{\infty} = (\sigma\sigma^*)_{ij}$ is strictly elliptic and bounded, $c < |X_s|^{-2} X_s^* \sigma\sigma^*(X_s) X_s < M$ for some $c > 0$ and $M < \infty$. Let $S_t = R_{A_t^{-1}}$. Then S_t is a semimartingale, $< S_t, S_t >= t$, and

$$dS_t = dW_t + (C_t/2S_t)dt$$

where W_t is a 1-dimensional Brownian Motion under P_x^{∞} and

$$C_t = \text{ trace } \sigma\sigma^*(X_t) - |X_t|^{-2} X_t^* \sigma\sigma^*(X_t) X_t.$$

But $|C_t|$ is uniformly bounded. Therefore by using a Girsanov transformation, there exists a probability measure P on $C[0, \infty)$ such that S_t is a standard Brownian motion up to the time S_t hits $1/2$. Since P_x^{∞} is equivalent to $P, \tau_{2+\epsilon} \longrightarrow \tau_2$ a.s., P_x^{∞}.

Hence τ_2 is a P_x^{∞}-continuous functional. Therefore we get

$$Q_n f(x) \longrightarrow Q_{\infty} f(x) \text{ on } S \text{ as } n \longrightarrow \infty.$$

On the other hand, by Krylov and Safonov [KS],

$$|Q_n f(x) - Q_n f(y)| \leq K |x - y|^{\gamma} \|f\| \tag{4}$$

where K and γ depend only on the ellipticity and bounds on a_{ij}^n. Since $a_{ij}^n \longrightarrow a_{ij}^{\infty}$ in measure, we can take a uniform bound of ellipticity for a_{ij}^n and a_{ij}^{∞}. Therefore there exist a single K and γ for which (4) holds for all $n = 1, 2, \cdots, \infty$. Hence $\{Q_n f, Q_{\infty} f\}$ forms an equicontinuous family. Therefore $Q_n f \longrightarrow Q_{\infty} f$ uniformly on S. Also Q_n and Q_{∞} are strongly positive by the support theorem ([SV]). Hence Q_n and Q_{∞} are strongly positive and compact operators on $C(S)$. Therefore by Krein-Rutman theorem, ρ_n and ρ, the frist eigenvalues of Q_n and Q_∞ respectively, are positive and also the corresponding eigenfunctions φ_n and φ are strictly positive ([KR] Theorems 6.1, 6.3 and the proof of Theorem 6.3). Moreover, if $f \in C(S)$

$$Q_n f(x) = \rho_n \mu_n(f)\varphi_n(x) + R_n f(x)$$

$$Q_\infty f(x) = \rho\mu(f)\varphi_n + Rf(x)$$

where μ, μ_n are eigenvectors for the adjoint operators Q^*_∞ and Q^*_n corresponding ρ and ρ_n and R_n, R are linear functionals on $C(S)$ with $\limsup_m \sqrt[m]{\|R_n^m\|} < \rho_n$ and $\limsup_m \sqrt[m]{\|R^m\|} < \rho$. Then by the analogue of Theorem 4.3 and 4.4 of [Kw], we have $\rho_n \longrightarrow \rho, \varphi_n \longrightarrow \varphi$ uniformly on $S, \mu_n(f_n) \longrightarrow \mu(f)$ and $R_n f_n \longrightarrow Rf$ where $\|f_n - f\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $h : R^d \longrightarrow R$ be continuous and 0 near 0 and P_x^0 denote the law of X_t under P_x killed when first reaching 0. That is

$$P_x^0(X_{t_1} \in A_1, \cdots, X_{t_n} \in A_n) = P_x\Big(\bigcap_{1 \le i \le n} \{X_{t_i} \in A_i, t_i < \tau_0\}\Big).$$

To show that P_x, the solution of $(MP)_x - L_a$, is unique, it suffices to show that for given h, $I(h)$ does not depend on the choice of the solutions $(MP)_x - L_a$ for $x = 0$ by Theorem 5.5 of [BP] and Theorem 4.7 of [Kw] where

$$I(h) = \frac{E^{P_0}[E^{P^0_{X_{\tau_\varepsilon}}} \int_0^{\tau_R} h(X_r)dt]}{E^{P_0}[P^0_{X_{\tau_\varepsilon}}(\tau_0 > \tau_R)]}$$

and $h(x) \equiv 0$ if $|x| < \varepsilon$.

Then by the analogue of the proof of Theorem 4.7 of [Kw], it can be proved that $I(h)$ does not depend on the choice of $(MP)_x - L_a$ and we have uniqueness of $(MP)_x - L_a$. Hence we get Theorem 2.1.

Suppose a_{ij} are as in Theorem 2.1. Suppose a_{ij}^n are smooth approximations to the a_{ij} and $a_{ij}^n \longrightarrow a_{ij}$ in the sense of Theorem 2.1. Then the solution of the martingale problem with respect to a_{ij}^n will be unique. Let P_x^n be the unique solution such that $P_x^n(X_0 = x) = 1$ and similarly define P_x with respect to a_{ij}. Then $P_x^n \longrightarrow P_x$.

Let $f \in C(S)$ and let h_n be a solution of the Dirichlet problem on the unit ball with continuous boundary fuction f for the operator L_n :

$$L_n = \sum_{i,j} \frac{1}{2} a_{ij}^n(x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

That is

$$L_n h_n = 0$$

on the interior of the unit ball and

$$h_n = f$$

on the boundary of the unit ball. Then h_n is unique and $h_n(x) = E^{P_x^n} f(X_{\tau_1})$. Therefore for x such that $0 <| x |< 1$, $h_n(x) \longrightarrow h(x) = E^{P_x} f(X_{\tau_1})$, since $P_x^n \longrightarrow P_x$ and τ_1 is P_x continuous by Lemma 3.1. Also, as the proof of Theorem 2.1 by Krylov and Safonov [KS],

$$|h_n(x) - h_n(y)| \leq K|x-y|^\gamma \|f\|$$

for $|x|, |y| < 1$ where K and γ depend on the ellipticity of a_{ij}^n. Since $a_{ij}^n \longrightarrow a_{ij}, h_n(x), h(x)$ are equicontinuous by the same argument as the proof of Theorem 2.1. Hence $h_n(x) \longrightarrow h(x)$ uniformly on compact subsets of the interior of the unit ball and we get Theorem 3.1, a result of Caffarelli [Ca] recently proved using partial differential equation techniques.

THEOREM 3.1. *Suppose the a_{ij} are as in Theorem 2.1. Suppose a_{ij}^n are smooth approximations to the a_{ij} with $a_{ij}^n \longrightarrow a_{ij}$ as in Theorem 2.1. Let $f \in C(S)$ and let h_n be the solution of the Dirichlet problem of the operator L_n on the unit ball with boundary function f. Then $h_n(x)$ converges uniformly on compact subsets of the interior of the unit ball to a function h wihch depends only on f and a_{ij} and does not depend on the approximation a_{ij}^n.*

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