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Small Discriminants

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Andrew Odlyzko found the following result about discriminants of number fields by using the ideas of Harold Stark. We will restrict ourselves to totally complex cases [5]. Let k be a totally complex number field with $[k : \mathbf{Q}] = n$, $D =$ the absolute value of the discriminant of k. Then

$$D^{1/n} \geq 21.8 \times e^{-70/n}.$$

Also we have

$$D^{1/n} \geq 41.6 \times e^{-3.7 \times 10^8/n}$$

if we assume the generalized Riemann hypothesis. Actually, Odlyzko found a quite impressive lower bound even for small integer n.

After this discovery of lower bounds by Stark and Odlyzko, some people searched for number fields with relatively small discriminants. Also some others studied algorithms which will produce number fields with the smallest absolute value of the discriminant for a fixed degree n. However, this algorithmic approach does not work well if the extension degree n is greater than seven or eight. And it is not clear whether current computers can finish those algorithms in a reasonable amount of time for large n. So we are justified in our approach for this subject. In this paper we will report the results of our computations. Some number fields we found are better than previously known examples. Others are rediscoveries of known examples. However, there is theoretically nothing new in our approach to this subject.

For a totally complex number field k, let d_k denote the absolute value of the discriminant of k. Let O_k denote the ring of algebraic integers of k, E_k the group of units of O_k, $Cl(k)$ the ideal class group of O_k. Let I

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be an ideal of O_k. Let k_I denote the ray class field of k with conductor I. Let

$$(1) \qquad O_k \longrightarrow O_k/I$$

denote the natural projection. E_k will be mapped onto $Im(E_k)$. Then we know [6] that the relative degree $h_I = [k_I : k]$ is equal to

$$h_I = \#\Big((O_k/I)^\times/Im(E_k)\Big) \cdot \#Cl(k).$$

We used this well-known formula to find number fields with relatively small discriminants. Let $[k : \mathbf{Q}] = n$. Let p be a prime number which splits completely in $k/\mathbf{Q}$. Let I denote a prime ideal of O_k which lies over p with residue class degree one. Let $k = \mathbf{Q}(\alpha)$. Then α is a root of an irreducible polynomial

$$f(X) \in \mathbf{Z}[X] \quad \text{of degree } n.$$

Since O_k/I is isomorphic to $\mathbf{Z}/p\mathbf{Z}$, each element of O_k is congruent to an integer modulo I. So there is an integer a so that $\alpha \equiv a \bmod I$. If we can express the generators of E_k by α, then it is easy to compute h_I from (1). All the computations can be done in $\mathbf{Z}/p\mathbf{Z}$. For the fields k we used, it turned out that $O_k = \mathbf{Z}[\alpha]$. Suppose that p is relatively prime to $h_I = [k_I : k]$. Then I is tamely ramified in k_I/k. Let d_I denote the absolute value of the discriminant of k_I. Then d_I is

$$d_I = (d_k)^{h_I} \cdot p^{e-1}$$

where

$$e = \#\Big((O_k/I)^\times/Im(E_k)\Big).$$

The first table was obtained by looking for ray class fields of totally complex quartic fields. In the table, $n = [k_I : \mathbf{Q}]$, $d =$ the absolute value of the discriminant of k, $f =$ the prime above the conductor I, $D =$ the absolute value of the discriminant of k_I, GRH means the lower bounds of $D^{1/n}$ obtained by assuming generalized Riemann hypothesis. The same notations apply to the second table too. All the fields used in computing these tables are given in [1], [2] where the generators of E_k are given too. The computations were carried out by a personal computer in two days. It is expected that, for example, the first line of the first table indicates the totally complex octic number field with smallest $D^{1/8}$. So trying to prove similar statements for various other n looks worthwhile.

Table by totally complex quartic fields

n	d	f	$D^{1/n}$	GRH
8	117	97	5.826	5.734
	125	101	5.953	
	144	73	5.923	
	189	61	6.198	
12	117	163	7.687	7.598
	125	151	7.716	
	189	127	8.313	
16	117	241	9.198	9.068
	125	521	10.805	
	144	193	9.292	
20	117	331	10.496	10.270
	189	181	10.487	
24	117	397	11.441	11.283
	125	541	12.406	
28	117	967	14.348	
	189	421	13.535	
44	144	661	15.155	14.728
	189	463	14.960	
48	198	937	17.748	15.225
52	125	911	16.114	15.680

Table by totally complex sectic fields

n	d	f	$D^{1/n}$	GRH
12	9747	733	8.009	7.598
	10051	557	7.868	
	10571	457	7.805	
	11691	337	7.737	
	12167	449	7.978	
	14283	229	7.747	
	14731	257	7.862	
18	9747	937	9.886	9.697
	10571	991	10.083	
	14731	571	10.023	
	16551	313	9.910	
	16807	673	10.435	
24	14731	937	11.646	11.283

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