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BOUNDED MULTIPLIER CONVERGENT SERIES AND ITS APPLICATIONS

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Using a matrix method, P. Antosik and C. Swartz have obtained a series of nice properties of bounded multiplier convergent (BMC) series on metric linear spaces ([1], [8], [9]). In this paper, we establish a basic property of BMC series on topological vector spaces which is a generalization of a result due to J. Batt ([2], Th. 2). From this, we have obtained a kind of inclusion theorem of operator spaces. This theorem yields a nice result on infinite systems of linear equations.

LEMMA 1. *Let X and Y be topological vector spaces and $f_j : Y \to X$ a linear operator for each $j \in N$. Then the following (1) and (2) are equivalent.*

(1) *The series $\sum_{j=1}^{\infty} f_j(y_j)$ converges for each bounded sequence $\{y_j\} \subseteq Y$.*

(2) *For every bounded subset B of Y, the series $\sum_{j=1}^{\infty} f_j(y_j)$ converges uniformly with respect to all sequences $\{y_j\} \subseteq B$.*

Proof. (2) $\to$ (1) is trivial.
(1) $\to$ (2). Suppose that (1) holds but (2) fails for some bounded subset B of Y. Then there is a neighborhood U of $0 \in X$ such that if $n_0 \in N$ then $\sum_{j=n}^{\infty} f_j(y_j) \notin U$ for some $n > n_0$ and some $\{y_j\} \subseteq B$. So there exist a positive integer $n_1 > 1$ and a sequence $\{y_{1j}\} \subseteq B$ such that $\sum_{j=n_1}^{\infty} f_j(y_{1j}) \notin U$. Pick a neighborhood V of $0 \in X$ for which $V + V \subseteq U$. Since the series $\sum_{j=n_1}^{\infty} f_j(y_{1j})$ converges by (1), we can find a

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positive integer $m_1 > n_1$ such that $\sum_{j=m_1+1}^{\infty} f_j(y_{1j}) \in V$. So $\sum_{j=n_1}^{m_1} f_j(y_j) \notin V$ because $\sum_{j=n_1}^{\infty} f_j(y_{1j}) \notin U$. Now, there exist a positive integer $n_2 > m_1$ and a sequence $\{y_{2j}\} \subseteq B$ such that $\sum_{j=n_2}^{\infty} f_j(y_{2j}) \notin U$. As above, we can find a positive integer $m_2 > n_2$ such that $\sum_{j=n_2}^{m_2} f_j(y_{2j}) \notin V$. Continuing this construction inductively give an integer sequence $n_1 < m_1 < n_2 < m_2 \cdots$ and a matix (y_{ij}) whose elements are in B such that $\sum_{j=n_i}^{m_i} f_j(y_{ij}) \notin V$ for each $i \in N$. Now set

$$y_j = \begin{cases} y_{ij}, & n_i \le j \le m_i,\ i = 1, 2, 3, \cdots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\{y_j\}$ is a bounded sequence in Y because $y_j = 0$ or $y_j \in B$, a bounded subset of Y. But the series $\sum_{j=1}^{\infty} f_j(y_j)$ diverges because $\sum_{j=n_i}^{m_i} f_j(y_j) = \sum_{j=n_i}^{m_i} f_j(y_{ij})$ for each $i \in N$ and $n_i \to +\infty, m_i \to +\infty$. This contradicts (1).

As a special case, we have

COROLLARY 2. *(J. Batt [2], Th.2). Let X and Y be normed spaces and (T_j) a sequence in $L(Y, X)$, the space of continuous linear operators. If the series $\sum_{j=1}^{\infty} T_j(y_j)$ converges for each bounded sequence $(y_j) \subseteq Y$, then $\sum_{j=1}^{\infty} T_j(y_j)$ converges uniformly with respcet to all sequences $(y_j) \subseteq B_Y = \{y \in Y : \|y\| \le 1\}$.*

The following useful conclusion also is a special case of Lemma 1.

COROLLARY 3. *Let X be a topological vector space and (x_j) a sequence in X. If the series $\sum_j x_j$ is BMC, i.e., the series $\sum_{j=1}^{\infty} t_j x_j$ converges* *for each* $(t_j) \in l_\infty$, *then* $\sum_{j=1}^{\infty} \alpha_j x_j$ *converges uniformly with respect to all sequences* $(\alpha_j) \in B_{l_\infty} = \{(\alpha_j) : \sup_j |\alpha_j| \leq 1\}$.

Proof. For each $j \in N$ define $f_j : \mathbf{C} \to X$ by $f_j(z) = zx_j, z \in \mathbf{C}$. Hence Lemma 1.

Denote that $D = \{z \in \mathbf{C} : \ |z| \leq 1\}$ and $D^N = \{(z_k) : z_k \in D$ for each $k\}$. The coordinatewise convergence topology for D^N is just the product topology for $D^N = D \times D \times D \times \cdots$. Since D is compact, by Tychonoff's theorem, D^N is compact with respect to the coordinatewise convergence topology. Note that, as abstract sets, $D^N = B_{l_\infty}$. We have the following important fact which is a new property of BMC series on topological vector spaces.

D. Randtke([7]) defines $\sum_j x_j$ in a locally convex space to be strongly summable if it is bounded multiplier convergent and $\{\sum_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1$, for all $j \in N\}$ is bounded. The following result shows that the latter condition is superfluous and, indeed, much more holds.

THEOREM 4. *Let* X *be a topological vector space and* (x_j) *a sequence in* X. *If the series* $\sum_j x_j$ *is BMC, then the set* $\{\sum_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1$, *for all* $j \in N\}$ *is a compact subset of* X.

Proof. For each $j \in N$ define $f_j : D \to X$ by $f_j(t) = tx_j, \ t \in D$. Then each f_j is continuous. Observe that the series $\sum_{j=1}^{\infty} f_j(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j$ converges for each $(\alpha_j) \in D^N$ since $\sum_j x_j$ is BMC, we have a function $F : D^N \to X$ such that $F((\alpha_j)) = \sum_{j=1}^{\infty} f_j(\alpha_j), \ (\alpha_j) \in D^N$. We need to show that F is continuous with respect to the coordinatewise convergence topology for D^N and the topology for X. Let $(t_j{}^{(\alpha)})_{\alpha \in I}$ be a convergent net in D^N, i.e., $\lim_\alpha t_j{}^{(\alpha)} = t_j \in D$ for each j. Let U be a neighborbood of $0 \in X$ and pick a neighborhood V of $0 \in X$

such that $V+V+V \subseteq U$. By Corollary 3, there is an $n_0 \in N$ such that $\sum\limits_{j=n_0+1}^{\infty} \alpha_j x_j \in V$ for all $(\alpha_j) \in D^N$ since $D^N = B_{l_\infty}$. Hence, $\sum\limits_{j=n_0+1}^{\infty} f_j(t_j) = \sum\limits_{j=n_0+1}^{\infty} t_j x_j \in V$ and $\sum\limits_{j=n_0+1}^{\infty} f_j(t_j{}^{(\alpha)}) = \sum\limits_{j=n_0+1}^{\infty} t_j{}^{(\alpha)} x_j \in V$ for all $\alpha \in I$. In the other hand, there is an $\alpha_0 \in I$ such that $\sum\limits_{j=1}^{n_0} t_j{}^{(\alpha)} x_j - \sum\limits_{j=1}^{n_0} t_j x_j = \sum\limits_{j=1}^{n_0} (t_j{}^{(\alpha)} - t_j) x_j \in V$ for all $\alpha \geq \alpha_0$ because $\lim\limits_{\alpha} t_j{}^{(\alpha)} = t_j$ for each $j \in N$.
Thus, if $\alpha \geq \alpha_0$ then we have

$$
\begin{aligned}
F((t_j{}^{(\alpha)})) - F((t_j)) &= \sum_{j=1}^{\infty} f_j(t_j{}^{(\alpha)}) - \sum_{j=1}^{\infty} f_j(t_j) \\
&= \sum_{j=1}^{n_0} [f_j(t_j{}^{(\alpha)}) - f_j(t_j)] + \sum_{j=n_0+1}^{\infty} f_j(t_j{}^{(\alpha)}) - \sum_{j=n_0+1}^{\infty} f_j(t_j) \\
&= \sum_{j=1}^{n_0} (t_j{}^{(\alpha)} - t_j) x_j + \sum_{j=n_0+1}^{\infty} t_j{}^{(\alpha)} x_j - \sum_{j=n_0+1}^{\infty} t_j x_j \in V+V+V \subseteq U,
\end{aligned}
$$

i.e., $\lim\limits_{\alpha} F((t_j{}^{(\alpha)})) = F((t_j))$. This shows that F is continuous and hence, $\{\sum\limits_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1, \text{ for all } j \in N\} = \{\sum\limits_{j=1}^{\infty} \alpha_j x_j : (\alpha_j) \in D^N\} = F[D^N]$ is compact.

Now let X be a Banach space and $K(l_\infty, X)$ the set of X-valued compact linear operators defined on l_∞. We would like to give some applications of Theorem 4 in the following Remarks.

REMARKS (1). Each continuous linear operator $\tau \in L(c_0, X)(L(l_p), X)$ with $p \geq 1)$, together with every $(t_j) \in c_0$ (l_p), determines a compact linear operator $A \in K(l_\infty, X) : A((\alpha_j)) = \sum\limits_{j=1}^{\infty} \alpha_j t_j \tau(e_j)$, $(\alpha_j) \in l_\infty$, where $\{e_j\}$ is the unit vector basis of $c_0(l_p)$.

We would like to represent above results as follows:
$c_0 \times L(c_0, X) \subseteq K(l_\infty, X)$, $l_p \times L(l_p, X) \subseteq K(l_\infty, X)$ for $p \geq 1$.

(2) Above inclusion result and the spectral theory of compact operators on Banach spaces ([4], Th.2.21) imply that:
Let (a_{ij}) be a matrix such that $\sup_i \sum_{j=1}^{\infty} |a_{ij}|^2 < +\infty$. Then there is a partition $\mathbf{C} = U \cup V$ $(U \cap V = \emptyset,\ U \neq \emptyset,\ V \neq \emptyset)$ such that for each $\lambda \in U$ the infinite system of linear equations

$$(*) \qquad \sum_{j=1}^{\infty} \frac{a_{ij}}{j} x_j - \lambda x_i = b_i, \ i = 1, 2, 3, \cdots$$

has a unique solution $(c_i) \in l_\infty$ for each $(b_i) \in l_\infty$ and, conversely, each $(c_i) \in l_\infty$ is just the unique solution of $(*)$ for some $(b_i) \in l_\infty$. And for each $\lambda \in V$ the homogeneous system $\sum_{j=1}^{\infty} \frac{a_{ij}}{j} x_j - \lambda x_i = 0,\ i = 1, 2, 3, \cdots$ has a nonzero solution. Moreover, V is countable and

$$U \supseteq \{\lambda \in \mathbf{C} : |\lambda| > \sup_i \sum_{j=1}^{\infty} \frac{|a_{ij}|}{j}\}.$$

This result is different from the classical result in [3], [5] and a recent result of author ([6], Corollary 3).

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