BOUNDDED MULTIPLIER CONVERGENT SERIES AND ITS APPLICATIONS

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Using a matrix method, P. Antosik and C. Swartz have obtained a series of nice properties of bounded multiplier convergent (BMC) series on metric linear spaces ([1], [8], [9]). In this paper, we establish a basic property of BMC series on topological vector spaces which is a generalization of a result due to J. Batt ([2], Th. 2). From this, we have obtained a kind of inclusion theorem of operator spaces. This theorem yields a nice result on infinite systems of linear equations.

**LEMMA 1.** Let $X$ and $Y$ be topological vector spaces and $f_j : Y \to X$ a linear operator for each $j \in \mathbb{N}$. Then the following (1) and (2) are equivalent.

1. The series $\sum_{j=1}^{\infty} f_j(y_j)$ converges for each bounded sequence $\{y_j\} \subseteq Y$.

2. For every bounded subset $B$ of $Y$, the series $\sum_{j=1}^{\infty} f_j(y_j)$ converges uniformly with respect to all sequences $\{y_j\} \subseteq B$.

**Proof.** (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Suppose that (1) holds but (2) fails for some bounded subset $B$ of $Y$. Then there is a neighborhood $U$ of $0 \in X$ such that if $n_0 \in \mathbb{N}$ then $\sum_{j=n_0}^{\infty} f_j(y_j) \notin U$ for some $n > n_0$ and some $\{y_j\} \subseteq B$. So there exist a positive integer $n_1 > 1$ and a sequence $\{y_{1j}\} \subseteq B$ such that $\sum_{j=n_1}^{\infty} f_j(y_{1j}) \notin U$. Pick a neighborhood $V$ of $0 \in X$ for which $V + V \subseteq U$. Since the series $\sum_{j=n_1}^{\infty} f_j(y_{1j})$ converges by (1), we can find a

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positive integer \( m_1 > n_1 \) such that \( \sum_{j=m_1+1}^{\infty} f_j(y_{1j}) \in V \). So \( \sum_{j=n_1}^{m_1} f_j(y_j) \notin V \) because \( \sum_{j=n_1}^{\infty} f_j(y_{1j}) \notin U \). Now, there exist a positive integer \( n_2 > m_1 \) and a sequence \( \{ y_{2j} \} \subseteq B \) such that \( \sum_{j=n_2}^{\infty} f_j(y_{2j}) \notin U \). As above, we can find a positive integer \( m_2 > n_2 \) such that \( \sum_{j=n_2}^{m_2} f_j(y_{2j}) \notin V \). Continuing this construction inductively give an integer sequence \( n_1 < m_1 < n_2 < m_2 \cdots \) and a matrix \( (y_{ij}) \) whose elements are in \( B \) such that \( \sum_{j=n_i}^{m_i} f_j(y_{ij}) \notin V \) for each \( i \in N \). Now set

\[
y_j = \begin{cases} 
y_{ij}, & n_i \leq j \leq m_i, \ i = 1, 2, 3, \cdots \\
0, & \text{otherwise.}
\end{cases}
\]

Then \( \{ y_j \} \) is a bounded sequence in \( Y \) because \( y_j = 0 \) or \( y_j \in B \), a bounded subset of \( Y \). But the series \( \sum_{j=1}^{\infty} f_j(y_j) \) diverges because \( \sum_{j=n_i}^{m_i} f_j(y_{ij}) = \sum_{j=n_i}^{m_i} f_j(y_{ij}) \) for each \( i \in N \) and \( n_i \to +\infty, m_i \to +\infty \). This contradicts (1).

As a special case, we have

**Corollary 2.** \((J.\ Batt [2], \text{Th.}2)\). Let \( X \) and \( Y \) be normed spaces and \((T_j)\) a sequence in \( L(Y, X) \), the space of continuous linear operators. If the series \( \sum_{j=1}^{\infty} T_j(y_j) \) converges for each bounded sequence \( \{ y_j \} \subseteq Y \), then \( \sum_{j=1}^{\infty} T_j(y_j) \) converges uniformly with respect to all sequences \( \{ y_j \} \subseteq B_Y = \{ y \in Y : \| y \| \leq 1 \} \).

The following useful conclusion also is a special case of Lemma 1.

**Corollary 3.** Let \( X \) be a topological vector space and \((x_j)\) a sequence in \( X \). If the series \( \sum_{j} x_j \) is BMC, i.e., the series \( \sum_{j=1}^{\infty} t_j x_j \) converges
for each \((t_j) \in l_\infty\), then \(\sum_{j=1}^{\infty} \alpha_j x_j\) converges uniformly with respect to all sequences \((\alpha_j) \in B_{l_\infty} = \{ (\alpha_j) : \sup_j |\alpha_j| \leq 1 \}\).

**Proof.** For each \(j \in \mathbb{N}\) define \(f_j : \mathbb{C} \to X\) by \(f_j(z) = zx_j, z \in \mathbb{C}\). Hence Lemma 1.

Denote that \(D = \{ z \in \mathbb{C} : |z| \leq 1 \}\) and \(D^N = \{ (z_k) : z_k \in D\) for each \(k\). The coordinatewise convergence topology for \(D^N\) is just the product topology for \(D^N = D \times D \times D \times \cdots\). Since \(D\) is compact, by Tychonoff's theorem, \(D^N\) is compact with respect to the coordinatewise convergence topology. Note that, as abstract sets, \(D^N = B_{l_\infty}\). We have the following important fact which is a new property of BMC series on topological vector spaces.

D. Randtke([7]) defines \(\sum_j x_j\) in a locally convex space to be strongly summable if it is bounded multiplier convergent and \(\{ \sum_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1, \text{ for all } j \in \mathbb{N} \}\) is bounded. The following result shows that the latter condition is superfluous and, indeed, much more holds.

**Theorem 4.** Let \(X\) be a topological vector space and \((x_j)\) a sequence in \(X\). If the series \(\sum_j x_j\) is BMC, then the set \(\{ \sum_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1, \text{ for all } j \in \mathbb{N} \}\) is a compact subset of \(X\).

**Proof.** For each \(j \in \mathbb{N}\) define \(f_j : D \to X\) by \(f_j(t) = tx_j, t \in D\). Then each \(f_j\) is continuous. Observe that the series \(\sum_{j=1}^{\infty} f_j(\alpha_j) = \sum_{j=1}^{\infty} \alpha_j x_j\) converges for each \((\alpha_j) \in D^N\) since \(\sum_j x_j\) is BMC, we have a function \(F : D^N \to X\) such that \(F((\alpha_j)) = \sum_{j=1}^{\infty} f_j(\alpha_j), (\alpha_j) \in D^N\). We need to show that \(F\) is continuous with respect to the coordinatewise convergence topology for \(D^N\) and the topology for \(X\). Let \((t_j^{(\alpha)})_{\alpha \in I}\) be a convergent net in \(D^N\), i.e., \(\lim\alpha \in I t_j^{(\alpha)} = t_j \in D\) for each \(j\). Let \(U\) be a neighborhood of \(0 \in X\) and pick a neighborhood \(V\) of \(0 \in X\)
such that $V + V + V \subseteq U$. By Corollary 3, there is an $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0+1}^{\infty} \alpha_j x_j \in V$ for all $(\alpha_j) \in D^N$ since $D^N = B_{l_{\infty}}$. Hence,

$$\sum_{j=n_0+1}^{\infty} f_j(t_j) = \sum_{j=n_0+1}^{\infty} t_j x_j \in V \quad \text{and} \quad \sum_{j=n_0+1}^{\infty} f_j(t_j^{(\alpha)}) = \sum_{j=n_0+1}^{\infty} t_j^{(\alpha)} x_j \in V \quad \text{for all } \alpha \in I.$$

In the other hand, there is an $\alpha_0 \in I$ such that

$$\sum_{j=1}^{n_0} t_j^{(\alpha)} x_j - \sum_{j=1}^{n_0} t_j x_j = \sum_{j=1}^{n_0} (t_j^{(\alpha)} - t_j) x_j \in V \quad \text{for all } \alpha \geq \alpha_0 \quad \text{because } \lim_{\alpha} t_j^{(\alpha)} = t_j \text{ for each } j \in \mathbb{N}.$$

Thus, if $\alpha \geq \alpha_0$ then we have

$$F((t_j^{(\alpha)})) - F((t_j)) = \sum_{j=1}^{\infty} f_j(t_j^{(\alpha)}) - \sum_{j=1}^{\infty} f_j(t_j)$$

$$= \sum_{j=1}^{n_0} [f_j(t_j^{(\alpha)}) - f_j(t_j)] + \sum_{j=n_0+1}^{\infty} f_j(t_j^{(\alpha)}) - \sum_{j=n_0+1}^{\infty} f_j(t_j)$$

$$= \sum_{j=1}^{n_0} (t_j^{(\alpha)} - t_j) x_j + \sum_{j=n_0+1}^{\infty} t_j^{(\alpha)} x_j - \sum_{j=n_0+1}^{\infty} t_j x_j \in V + V + V \subseteq U,$$

i.e., $\lim_{\alpha} F((t_j^{(\alpha)})) = F((t_j))$. This shows that $F$ is continuous and hence, \( \{ \sum_{j=1}^{\infty} \alpha_j x_j : |\alpha_j| \leq 1, \text{ for all } j \in \mathbb{N} \} = \{ \sum_{j=1}^{\infty} \alpha_j x_j : (\alpha_j) \in D^N \} = F[D^N] \) is compact.

Now let $X$ be a Banach space and $K(l_{\infty}, X)$ the set of $X$-valued compact linear operators defined on $l_{\infty}$. We would like to give some applications of Theorem 4 in the following Remarks.

**Remarks (1).** Each continuous linear operator $\tau \in L(c_0, X)(L(l_p), X)$ with $p \geq 1$, together with every $(t_j) \in c_0$ ($l_p$), determines a compact linear operator $A \in K(l_{\infty}, X)$ : $A((\alpha_j)) = \sum_{j=1}^{\infty} \alpha_j t_j \tau(e_j)$, $(\alpha_j) \in l_{\infty}$, where $\{e_j\}$ is the unit vector basis of $c_0(l_p)$.

We would like to represent above results as follows:

$c_0 \times L(c_0, X) \subseteq K(l_{\infty}, X)$, $l_p \times L(l_p, X) \subseteq K(l_{\infty}, X)$ for $p \geq 1$. 

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(2) Above inclusion result and the spectral theory of compact operators on Banach spaces ([4], Th.2.21) imply that:

Let \((a_{ij})\) be a matrix such that \(\sup_{i,j} \sum_{j=1}^{\infty} |a_{ij}|^2 < +\infty\). Then there is a partition \(C = U \cup V (U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset)\) such that for each \(\lambda \in U\) the infinite system of linear equations

\[
(*) \sum_{j=1}^{\infty} a_{ij} x_j - \lambda x_i = b_i, \ i = 1, 2, 3, \ldots
\]

has a unique solution \((c_i) \in l_\infty\) for each \((b_i) \in l_\infty\) and, conversely, each \((c_i) \in l_\infty\) is just the unique solution of \((*)\) for some \((b_i) \in l_\infty\). And for each \(\lambda \in V\) the homogeneous system \(\sum_{j=1}^{\infty} \frac{a_{ij}}{j} x_j - \lambda x_i = 0, \ i = 1, 2, 3, \ldots\) has a nonzero solution. Moreover, \(V\) is countable and

\[
U \supseteq \{ \lambda \in C : |\lambda| > \sup_{i,j} \sum_{j=1}^{\infty} \frac{|a_{ij}|}{j} \}.
\]

This result is different from the classical result in [3], [5] and a recent result of author ([6], Corollary 3).

References

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