

## COMMUTING INVOLUTIONS IN A LEFT ARTINIAN RING

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The involutions in a left Artinian ring  $A$  with identity are investigated. Those left Artinian rings  $A$  for which  $2$  is a unit in  $A$  and the set of involutions in  $A$  forms a finite abelian group are characterized by the number of involutions in  $A$ .

### 1. Introduction and basic definitions

Let  $A$  be a left Artinian ring with identity  $1$ , let  $G$  denote the group of all units in  $A$  and let  $J$  denote the Jacobson radical of  $A$ . An involution in  $A$  is an element  $g$  in  $G$  such that  $g^2 = 1$ . An idempotent in  $A$  is an element  $e$  in  $A$  such that  $e^2 = e$ . Note that if  $2$  is a unit in  $A$ , then the mapping  $e \rightarrow 1 - 2e$  is a bijection from the set of idempotents of  $A$  to the set of involutions of  $A$ .

Consequently, if  $2$  is a unit in  $A$ , then  $ef = fe$  for all idempotents  $e$  and  $f$  in  $A$  if and only if  $g_1g_2 = g_2g_1$  for all involutions  $g_1$  and  $g_2$  in  $A$ . Clearly,  $g_1g_2 = g_2g_1$  for involutions  $g_1$  and  $g_2$  in  $A$  if and only if the set  $\Delta$  of all involutions in  $A$  forms an abelian group under multiplication.

In [3], Cohen and Koh proved that if any ring with  $1$  has finite number of involutions in  $A$ , then the number is  $1$  or even, Wedderburn - Artinian proved that if  $A$  is a semisimple Artinian ring, then  $A$  is isomorphic to a direct product of a finite number of simple rings. Hence we obtain the following:

**THEOREM 1.1.** *If  $A$  is a left Artinian ring with identity, then  $A/J \cong \bigoplus_{i=1}^n M_i(D_i)$  where  $M_i(D_i)$  is the set of all the  $n_i \times n_i$  matrices over a division ring  $D_i$  for each  $i = 1, 2, \dots, n$  and  $n$  is a positive integer.*

## 2. Commuting involutions in a left Artinian ring

Recall that a ring  $A$  is said to be primitive if it has a faithful, irreducible  $A$ -module and an ideal  $P$  in a ring  $A$  is said to be primitive if  $A/P$  is a primitive ring. By the well-known fact that if a ring  $A$  has identity, then the  $A$ -module  $M$  is irreducible if and only if there exists a maximal left ideal  $I$  such that  $A/I$  and  $M$  are isomorphic, in [2], the following theorem was proved:

**THEOREM 2.1.** *If a ring  $A$  has identity, then  $(I : A) = \{a \in A : aA \subseteq I\}$  is a primitive ideal if and only if  $I$  is a maximal. And if  $I$  is a left ideal of  $A$ , then  $(I : A)$  is the largest ideal of  $A$  contained in  $A$ .*

*Proof.* In [2], (pp 52, Theorem 4)

In [2], the following theorem was also proved:

**THEOREM 2.2.** *A primitive ideal of a ring  $A$  is prime.*

*Proof.* In [2], (pp 67, Theorem 6).

**LEMMA 2.3.** *Let  $A$  be a left Artinian ring with identity such that  $A/J \cong \bigoplus_{i=1}^n D_i$  where each  $D_i$  is a division ring of odd characteristic. If the set  $\Delta$  of involutions in  $A$  forms an abelian group, then  $A$  has precisely  $2^n$  involutions and  $n$  maximal left (or right) ideals.*

*Proof.* Since  $2 + J$  is a unit in  $A/J$ ,  $2$  is a unit in  $A$ . Therefore, the number of involutions in  $A$  is equal to the number of idempotents in  $A$ . By hypothesis,  $A/J$  has precisely  $2^n$  idempotents, namely,  $\{\bigoplus_{i=1}^n x_i \mid x_i = 0_i, 1_i\}$  where  $0_i$  (resp.  $1_i$ ) is the additive identity (resp. multiplicative identity) of  $D_i$ .

Since each idempotent of  $A/J$  may be lifted to an idempotent of  $A$ ,  $A$  has at least  $2^n$  idempotents. Suppose that  $|\Delta| > 2^n$ . Then there exist distinct idempotents  $e$  and  $f$  in  $A$  such that  $e + J = f + J$ . Thus  $e - f \in J$ . Since  $J$  is a nilpotent ideal of  $A$ , there exists a positive integer  $m$  such that  $(e - f)^m = 0$  ( $m \geq 2$ ). Note that if  $m$  is odd, then  $(e - f)^m = e - f$  and if  $m$  is even, then  $e - 2ef + f = 0$ , and so  $e + f = 2ef - (*)$ . By multiplying  $e$  to both sides of  $(*)$ , we get  $e = ef$ . Thus  $e + f = 2ef = 2e$  and so  $e = f$ , a contradiction. Hence  $A$  has precisely  $2^n$  idempotents and hence  $2^n$  involutions.

Next, for the simplicity of notion, assume that  $A/J = \bigoplus_{i=1}^n D_i$  and let  $\Pi$  denote the canonical epimorphism of  $A$  onto  $A/J$ . For each  $j = 1, 2, \dots, n$ , let  $A_j = \Pi^{-1}(\bigoplus_{i=1}^n B_i)$  where  $B_j = \{0_j\}$  and  $B_i = D_i$  for  $i \neq j$ . Note that  $A_1, A_2, \dots, A_n$  are maximal left (or right) ideals of  $A$ . Moreover, if  $I$  is any maximal left (or right) ideal of  $A$ , then  $A_1 A_2 \cdots A_n \subseteq \bigcap_{i=1}^n A_i \subseteq J \subseteq I$ . By theorem 2.1,  $(I : A)$  is primitive and the largest ideal contained in  $A$  and so  $A_1 A_2 \cdots A_n \subseteq (I : A)$ . By theorem 2.2,  $(I : A)$  is a prime ideal. Thus there exists an  $i, 1 \leq i \leq n$ , such that  $A_i \subseteq I$ . Consequently,  $A_i = I$ . So  $A$  has precisely  $n$  maximal left (or right) ideals.

**THEOREM 2.4.** *Let  $A$  be a left Artinian ring with identity such that  $2 \cdot 1$  is a unit in  $A$ , the set  $\Delta$  of involutions in  $A$  is an abelian group under multiplication and  $|\Delta|$  is finite. Let  $m = |\Delta|$ . Then  $m = 2^n$  for some positive integer  $n$  and the following are equivalent:*

- (i)  $m = 2^n$ .
- (ii)  $A/J \cong \bigoplus_{i=1}^n D_i$  where each  $D_i$  is a division ring of odd characteristic.
- (iii)  $A$  has precisely  $n$  maximal left (or right) ideals.

*Proof.* Since the order of each  $g$  in  $A$  divides 2,  $|\Delta| = 2^n$  for some  $n \geq 0$ . Since  $2 \cdot 1$  is a unit in  $A$ ,  $\text{char}(A) \neq 2$  and so  $n > 0$ . By the preceding Lemma, (ii) implies (i) and (ii) implies (iii).

Assume (i) holds. Note that  $A/J \cong \bigoplus_{i=1}^n M_i(D_i)$  where each  $M_i(D_i)$  is the set of all the  $n_i \times n_i$  matrices over a division ring  $D_i$ . Since the idempotents in  $A$  commute and for each idempotent  $\bar{e}$  in  $A/J$ , there exists an idempotents  $e$  in  $A$  such that  $e + J = \bar{e}$  and if  $\bar{e}_{ni}$  and  $\bar{f}_{ni}$  are idempotents in  $M_i(D_i)$ , then  $\bar{e}_{ni} \bar{f}_{ni} = \bar{f}_{ni} \bar{e}_{ni}$ . So  $n_i = 1$  for all  $i = 1, 2, \dots, k$ , that is,  $A/J \cong \bigoplus_{i=1}^n D_i$ . Since  $2 \cdot 1$  is a unit in  $A$ ,  $2 \cdot 1 + J$  is a unit in  $A/J$ . Therefore, each  $D_i$  is a division ring of odd characteristic. By Lemma 2.3,  $k = n$ . Hence (i) implies (ii).

Finally, assume (iii) holds. As above,  $A/J \cong \bigoplus_{i=1}^n D_i$  where each  $D_i$  is a division ring of odd characteristic. Once again, Lemma 2.3 yields

that  $k = n$ .

**COROLLARY 2.5.** *Let  $A$  be a commutative Artinian ring with identity such that  $2 \cdot 1$  is a unit in  $A$ , the set  $\Delta$  of involutions in  $A$  is an abelian group under multiplication and  $|\Delta|$  is finite. Let  $m = |\Delta|$ . Then  $m = 2^n$  for some positive integer  $n$  and the following are equivalent:*

- (i)  $m = 2^n$
- (ii)  $A/J \cong \bigoplus_{i=1}^n F_i$  where each  $F_i$  is a field of odd characteristic.
- (iii)  $A$  has precisely  $n$  maximal ideals.
- (iv)  $A \cong \bigoplus_{i=1}^n A_i$  where each  $A_i$  is an Artinian local ring with identity such that  $(A_i/J_i)$  is odd where for each  $i$ ,  $J_i$  is the Jacobson radical of  $A_i$ .

*Proof.* By [4, Theorem IV.2.9], if  $A_i$  is an Artinian local ring with identity and  $\text{char}(A_i/J_i)$  is odd, then  $A_i$  has exactly 2 involutions. Therefore (iv) implies (i). Since  $A$  is a commutative Artinian ring with identity and any Artinian ring is semiperfect, by [1, Proposition 7.6, and Theorem 27.6],  $A \cong \bigoplus_{i=1}^n A_i$  where each  $A_i$  is an Artinian local ring with identity. Since  $2 \cdot 1$  is a unit in  $A$ ,  $\text{char}(A_i/J_i)$  is odd for all  $i = 1, 2, \dots, m$ . Thus  $A_i$  has exactly 2 involutions for all  $i$  [Theorem IV.2.9]. Since the number of involutions in  $\bigoplus A_i$  is  $2^n$ ,  $m = n$  if (i) holds. So (i) implies (iv). Therefore (i) – (iv) are equivalent by Theorem 2.4.

As noted previously, the assumption that the set of all involutions in a ring  $A$  is equivalent to the property that  $ef = fe$  for all idempotents  $e$  and  $f$  in  $A$  where  $2$  is a unit in  $A$ . We next classify left Artinian rings  $A$  with identity for which the idempotents of  $A$  commute. For the simplicity of notation, assume  $A/J = \bigoplus_{i=1}^n M_i(D_i)$  where each  $M_i(D_i)$  is the set of all the  $n_i \times n_i$  matrices over a division ring  $D_i$ . Let  $\Phi$  denote the canonical homomorphism of  $A$  onto  $A/J$ . For each  $i$ , let  $M_j = \bigoplus_{i=1}^n B_i$  where  $B_j = M_j$  and  $B_i = \{0_i\}$  for  $i \neq j$  and let  $A_j = \Phi^{-1}(M_j)$ .

**COROLLARY 2.6.** *Let  $A$  be a left Artinian ring with identity such that  $2$  is a unit in  $A$  and  $A$  has precisely four involutions. Then  $A$  has*

precisely two maximal left ideals and  $A/J \cong F_1 \times F_2$  where  $F_i$  is a field of odd characteristic.

*Proof.* Since  $A$  has precisely four involutions and 2 is a unit in  $A$ ,  $A$  has exactly four idempotents,  $0, 1, e$  and  $1 - e$  for some  $e$  in  $A$ . Since these idempotents commute, the involutions commute as well. Thus the corollary follows from Theorem 2.4.

**THEOREM 2.7.** *Let  $A$  be a left Artinian ring with identity. The following are equivalent:*

- (i) *For all idempotents  $e$  and  $f$  in  $A$ ,  $ef = fe$ .*
- (ii)  *$A/J \cong \bigoplus_{i=1}^n D_i$  where each  $D_i$  is a division ring and  $n$  is a positive integer,  $A_i$  contains a unique nonzero idempotent  $\varepsilon_j$  such that if  $e$  is any idempotent in  $A$ , then  $\varepsilon_j e = \varepsilon_j$  or  $e \varepsilon_j = \varepsilon_j e = 0$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Assume that (i) holds. Then if  $e$  and  $f$  are idempotents in  $A/J$ , then  $ef = fe$ . Consequently,  $A/J \cong \bigoplus_{i=1}^n D_i$  where each  $D_i$  is a division ring. We may assume that  $A/J = \bigoplus_{i=1}^n D_i$ .

Let  $e_j = (x_1, x_2, \dots, x_n)$  where  $x_j = 1_j$  and  $x_i = 0_i$  for all  $i \neq j$  and let  $\varepsilon_j$  be an idempotent in  $A$  such that  $\varepsilon_j + J = e_j$ . Then  $\varepsilon_j$  is a nonzero idempotent in  $A_j$ . For, let  $f \in A_j$  be such that  $f^2 = f$ . Then  $f + J = J$  or  $f + J = \varepsilon_j + J$  since the only idempotents in  $D_j$  are  $1_j$  and  $0_j$ . If  $f + J = J$ , then  $f = 0$  since  $J$  contains no nonzero idempotent. If  $f + J = \varepsilon_j + J$ , then  $f - \varepsilon_j \in J$  since idempotents in  $A$  commute,  $f - \varepsilon_j$  is idempotent of  $A$  and hence  $f = \varepsilon_j$ . Now let  $\varepsilon$  be an idempotent in  $A$ . Then  $\varepsilon_j e$  is an idempotent in  $A$ , and so  $\varepsilon_j e = 0 = e \varepsilon_j$  or  $\varepsilon_j e = \varepsilon_j$ .

(ii)  $\Rightarrow$  (i). Assume that (ii) holds. Let  $e$  and  $f$  be idempotents in  $A$ . We first show that  $\varepsilon_i(ef - fe) = 0$  for each  $i = 1, 2, \dots, n$ . By assumption,  $\varepsilon_i e = 0 = e \varepsilon_i$  or  $\varepsilon_i e = \varepsilon_i$ . If  $\varepsilon_i e = 0$ , then  $\varepsilon_i(ef - fe) = \varepsilon_i f e = 0$  since  $\varepsilon_i f = \varepsilon_i$  or  $\varepsilon_i f = 0$ . Suppose that  $\varepsilon_i e = \varepsilon_i$ . If  $\varepsilon_i f = 0$ , then as above,  $\varepsilon_i(ef - fe) = 0$ . Therefore, we may assume that  $\varepsilon_i e = \varepsilon_i = \varepsilon_i f$ . Then  $\varepsilon_i(ef - fe) = \varepsilon_i f - \varepsilon_i e = \varepsilon_i - \varepsilon_i = 0$ . Next note that for all  $i, j, i \neq j, \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i = 0$ . For, if not, then  $\varepsilon_i \varepsilon_j = \varepsilon_i$  and  $\varepsilon_j \varepsilon_i = \varepsilon_i$  so  $\varepsilon_i \varepsilon_j \varepsilon_i \varepsilon_j = \varepsilon_i \varepsilon_j$  which means that  $\varepsilon_i \varepsilon_j$  is idempotent which is contained in  $J$ . Since  $J$  contains no nonzero

idempotent,  $\varepsilon_i \varepsilon_j = 0$ , a contradiction. So  $\{\varepsilon_i : i = 1, 2, \dots, n\}$  is a set of orthogonal idempotents in  $A$ . Note that for  $j = 1, 2, \dots, n$ ,  $\varepsilon_j + J = (0_1, \dots, 1_j, \dots, 0_n)$  and hence  $\bigoplus_{i=1}^n \varepsilon_j + J = 1 + J$ . So  $1 - \bigoplus_{j=1}^n \varepsilon_j \in J$ . Since  $\{\varepsilon_i : i = 1, 2, \dots, n\}$  is a set of orthogonal idempotents in  $A$ ,  $1 - \bigoplus \varepsilon_j$  is also idempotent. And again,  $1 - \bigoplus_{j=1}^n \varepsilon_j = 0$ , so  $1 = \bigoplus_{j=1}^n \varepsilon_j$ . Therefore,  $(ef - fe) = 1(ef - fe) = (\bigoplus_{j=1}^n \varepsilon_j)(ef - fe) = \bigoplus_{j=1}^n (\varepsilon_j(ef - fe)) = 0$ . Thus, (ii) implies (i).

### References

1. Anderson, F. W. and Fuller, K. R., *Rings and Categories of Modules*, New York Springer-Verlag, 1974.
2. Behrens, E. A., *Ring Theory*, New York and London; Academic Press, 1972.
3. Cohen, J. and Koh, K., *Involutions in a compact ring*, Journal of Pure and Applied Algebra, **59**(1989), 151-168.
4. Han, J. C., *Involution in a left artinian ring*, Pro. of Kaist mathematics workshop, Vol. **5**(1990), 175-196.
5. Herstein, I. N., *Noncommutative Rings*, The Mathematical Association of America, 1965.
6. Lambek, J. *Lectures on Rings and Modules*, New York: Chelsea publishing Company., 1976.

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