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ON THE SPECTRAL RIGIDITY OF ALMOST ISOSPECTRAL MANIFOLDS

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1. Introduction

Let (M, g, J) be a closed Kähler manifold of complex dimension $m > 1$. We denote by $Spec(M, g)$ the spectrum of the real Laplace-Beltrami operator Δ acting on functions on M. The following characterization problem on the spectral rigidity of the complex projective space (CP^m, g_0, J_0) with the standard complex structure J_0 and the Fubini-Study metric g_0 has been attacked by many mathematicians : if (M, g, J) and (CP^m, g_0, J_0) are isospectral then is it true that (M, g, J) is holomorphically isometric to (CP^m, g_0, J_0) ?

In [BGM], [LB], it is proved that if (M, J) is (CP^m, J_0) then the answer to the problem is affirmative. Tanno ([Ta]) has proved that the answer is affirmative if $m \leq 6$. Recently, Wu ([Wu]) has showed in a more general sense that if (M, g) and (CP^m, g_0) are $(-\frac{4}{m})$-isospectral, $m \geq 4$, and if the second betti number $b_2(M)$ is equal to $b_2(CP^m)$ then the answer is affirmative.

The main purpose of this paper is to investigate the isospectral problem of CP^m without the assumption that $b_2(M) = b_2(CP^m)$. Then we shall prove that the answer to the problem is yes in the following cases:

(A) (M, g) and (CP^m, g_0) are (-1)-isospectral and $4 \leq m \leq 6$,
(B) (M, g) and (CP^m, g_0) are $(-\frac{4}{m})$-isospectral, $m \geq 4$ and $B_0 = B$.

2. Almost isospectral manifolds

Let (M, g) be a closed Riemannian manifold of dimension m. The Laplace-Beltrami operator Δ acting on functions on M has a discrete

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spectrum $Spec(M,g) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \nearrow \infty\}$. Consider the heat operator $e^{-t\Delta}$ given by

$$e^{-t\Delta}f(x) = \int_M K(t,x,y)f(y)\mu_M(y),$$

where $K(t,x,y) \in \mathrm{Hom}(T_yM, T_xM)$ is the kernel function. We have the well-defined asymptotic expansion for the L^2-trace of $e^{-t\Delta}$ for $t \downarrow 0$;

$$\mathrm{Tr}e^{-t\Delta} = \sum_{n=0}^{\infty} e^{-t\lambda_n} \overset{t\downarrow 0}{\sim} (4\pi t)^{-\frac{m}{2}} \sum_{n=0}^{\infty} t^n a_n,$$

where $a_n := \int_M a_n(x,\Delta)\mu_M(x)$ are the spectral invariants of Δ depending only on the discrete spectrum $Spec(M,g)$.

Let $R := (R^i_{jkl})$, $\rho := (R_{ij}) = (\sum R^k_{ikj})$ and $s := \sum R_{ii}$, $i,j,k,l = 1,...,m$ denote its Riemannian curvature tensor, Ricci curvature tensor and the scalar curvature of the Levi-Civita connection ∇ of M, respectively. It is well-known ([BGM],[Sa]) that

$$a_0 = vol(M), \tag{1}$$

$$a_1 = \frac{1}{6}\int_M s\mu_M, \tag{2}$$

$$a_2 = \frac{1}{360}\int_M \{2|R|^2 - 2|\rho|^2 + 5s^2\}\mu_M, \tag{3}$$

$$\begin{aligned} a_3 = \frac{1}{6!}\int_M \Big\{ & -\frac{1}{9}|\nabla R|^2 - \frac{26}{63}|\nabla\rho|^2 - \frac{142}{63}|\nabla s|^2 \\ & -\frac{8}{21}R^{ij}{}_{kl}R^{kl}{}_{rs}R^{rs}{}_{ij} - \frac{8}{63}R^{rs}R_r{}^{jkl}R_{sjkl} + \frac{2}{3}s|R|^2 \\ & -\frac{20}{63}R^{ik}R^{jl}R_{ijkl} - \frac{4}{7}R^i{}_jR^j{}_kR^k{}_i - \frac{2}{3}s|\rho|^2 + \frac{5}{9}s^3 \Big\}\mu_M. \end{aligned} \tag{4}$$

DEFINITION. We say that two Riemannian manifolds (M,g) and $(\bar{M},\bar{g})$ are isospectral if $Spec(M,g) = Spec(\bar{M},\bar{g})$, and more generally α-isospectral if $\limsup_{n\to\infty}|\lambda_n - \bar{\lambda}_n|n^{-\alpha} = C < \infty$.

LEMMA(cf.[Wu]). *Let (M,g) and $(\bar{M},\bar{g})$ be two closed α-isospectral Riemannian m-manifolds.*

(i) *If $\alpha=-\frac{4}{m}$ and $m\geq 4$ then $a_i=\bar{a}_i$, $i=0,1,2$.*
(ii) *If $\alpha=-1$ then $a_i=\bar{a}_i$ for all $i\leq[\frac{m}{2}]$.*

Proof. (i) is proved in [Wu]. Further, we can prove (ii) by a similar argument of [Wu].

3. The conformal curvature tensor on a Hermitian manifold

Let (M,g) be a Hermitian manifold of complex dimension $m\geq 3$. Let R be its curvature tensor of the Hermitian connection of M. In terms of local complex coordinates $(z^1,...,z^m)$, we adopt ranges of indices : $i,j,k,\dots=1,...,m, i^*=i+m$, and $A,B,C,\dots=1,...,m,\ 1^*,...,m^*$.

We set $Z_i:=\frac{\partial}{\partial z^i}$, $Z_{i^*}:=\frac{\partial}{\partial\bar{z}^i}$ and we define

$$R(Z_C,Z_D)Z_B:=\sum R^A{}_{BCD}Z_A,\quad R_{ABCD}:=g(R(Z_C,Z_D)Z_B,Z_A).$$

Then we have the curvature tensor B_0 which is invariant under the conformal change of the Hermitian metrics ([KMP]);

(5)

$$\begin{aligned}B_{0,ij^*kl^*}&=R_{ij^*kl^*}+\frac{1}{m}(g_{ij^*}T_{kl^*}+S_{ij^*}g_{kl^*})\\&\quad-\frac{mr+(m^2-2)s}{2m^2(m^2-1)}g_{ij^*}g_{kl^*}+\frac{mr-s}{2m(m^2-1)}g_{il^*}g_{kj^*},\end{aligned}$$

where (R_{ij^*}), (S_{ij^*}), (T_{ij^*}) are distinct Ricci curvature tensors locally given by

$$R_{ij^*}:=-\sum g^{kl^*}R_{il^*kj^*},\ S_{ij^*}:=-\sum g^{kl^*}R_{ij^*kl^*},$$
$$T_{ij^*}:=-\sum g^{kl^*}R_{kl^*ij^*},$$

and r, s, t distinct scalar curvatures by

$$r:=2\sum R_{ii^*},\ s:=2\sum S_{ii^*},\ t:=2\sum T_{ii^*}.$$

In particular, if (M, g) is Kähler, then B_0 is written locally as followings;

$$
\begin{aligned}
B_{0,ij^*kl^*} &= R_{ij^*kl^*} + \frac{1}{m}(g_{ij^*}R_{kl^*} + R_{ij^*}g_{kl^*}) \\
&\quad - \frac{(m+2)s}{2m^2(m+1)} g_{ij^*}g_{kl^*} + \frac{s}{2m(m+1)} g_{il^*}g_{kj^*}.
\end{aligned} \tag{6}
$$

Moreover, this tensor vanishes identically if and only if the Kähler metric has constant holomorphic sectional curvature ([KMP]).

4. Isospectral rigidity of CP^m

Let (M, g) be a closed Kähler manifold of complex dimension $m \geq 2$. In terms of local complex coordinates, the Riemannian curvature tensor R, the Ricci curvature tensor ρ and the scalar curvature s satisfy $|R|^2 = 4\sum |R^i{}_{jkl^*}|^2$, $|\rho|^2 = 2\sum |R_{ij^*}|^2$ and $s = 2\sum R_{ii^*}$.

Let B be the Bochner curvature tensor on (M, g) given by

$$
\begin{aligned}
B_{ij^*kl^*} :&= R_{ij^*kl^*} + \frac{1}{m+2}\{g_{ij^*}R_{kl^*} + R_{ij^*}g_{kl^*} + g_{il^*}R_{kj^*} + R_{il^*}g_{kj^*}\} \\
&\quad - \frac{s}{2(m+1)(m+2)}\{g_{ij^*}g_{kl^*} + g_{il^*}g_{kj^*}\}.
\end{aligned} \tag{7}
$$

Then we have

$$
|B|^2 = |R|^2 - \frac{8}{m+2}|\rho|^2 + \frac{2}{(m+1)(m+2)}s^2. \tag{8}
$$

Note that a Kähler manifold (M, g) is constant holomorphic sectional curvature if and only if (M, g) is Bochner flat ($B = 0$) and Einstein.

Furthermore, we have

$$
|\rho|^2 \geq \frac{s^2}{2m}, \tag{9}
$$

the equality sign valids if and only if (M, g) is Einstein.

THEOREM A. *For $4 \le m \le 6$, if a closed Kähler manifold (M, g) and (CP^m, g_0) are (-1)-isospectral then (M, g) is holomorphically isometric to (CP^m, g_0).*

Proof. Let $(\bar{M}, \bar{g})$ denote the complex projective space (CP^m, g_0). By the Lemma, for $m = 4$ or 5 we have $a_i = \bar{a}_i$, $i = 0, 1, 2$. Since $(\bar{M}, \bar{g})$ is of constant holomorphic sectional curvature, (3) and (8) imply

$$(10) \quad \int_M \left[2|B|^2 + \frac{2(6-m)}{m+2} \left(|\rho|^2 - \frac{s^2}{2m} \right) \right] \mu_M + \frac{5m^2+4m+3}{m(m+1)} \left\{ \int_M s^2 \mu_M - \int_{\bar{M}} \bar{s}^2 \mu_{\bar{M}} \right\} = 0.$$

Moreover, the Schwarz inequality yields

$$(11) \qquad \int_M s^2 \mu_M \ge \int_{\bar{M}} \bar{s}^2 \mu_{\bar{M}}.$$

Thus it follows from (9),(10) and (11) that if $m = 4$ or 5 then $B = 0$, $|\rho|^2 = \frac{s^2}{2m}$ and $s = \bar{s}$. Therefore (M, g) is of constant holomorphic sectional curvature.

Now, let $m = 6$. Then $a_i = \bar{a}_i$, $i = 0, 1, 2, 3$. This, together with (10), gives that $B = 0$ and $s = \bar{s} =$ constant. By a similar way as in the proof of Tanno ([Ta]), we compute a_3 :

(12)

$$a_3 = \frac{1}{6!} \int_M \left\{ \frac{32(3m^2+20m+16)}{63m(m+1)(m+2)^2} - \frac{8(2m+1)}{63m(m+1)} + \frac{2(6-m)}{3(m+2)} \right\} s|\rho|^2 \mu_M + \frac{1}{6!} \int_M \left\{ \frac{35m^2+35m+4}{63m(m+1)} + \frac{4(4-21m)}{63m(m+1)(m+2)} - \frac{16(m^2+6m+6)}{21m(m+1)^2(m+2)^2} \right\} s^3 \mu_M.$$

Thus from $a_3 = \bar{a}_3$ and (12), it follows that

$$\frac{1}{147} \int_M s \left(|\rho|^2 - \frac{s^2}{12} \right) \mu_M - \frac{1}{147} \int_{\bar{M}} \bar{s} \left(|\bar{\rho}|^2 - \frac{\bar{s}^2}{12} \right) \mu_{\bar{M}} + \frac{3281}{6174} \left[\int_M s^3 \mu_M - \int_{\bar{M}} \bar{s}^3 \mu_{\bar{M}} \right] = 0,$$

which implies that $|\rho|^2 = \frac{s^2}{12}$. Since $B = 0$, this means that (M, g) is of constant holomorphic sectional curvature. Therefore, by the characterization of complex manifolds with constant holomorphic sectional curvature ([BGM]), for $4 \leq m \leq 6$ (M, g) is holomorphically isometric to (CP^m, g_0).

THEOREM B. *For $m \geq 4$, if a closed Kähler manifold (M, g) and (CP^m, g_0) are $(-\frac{4}{m})$-isospectral and if the conformal curvature tensor B_0 coincides with the Bochner curvature tensor B, then (M, g) is holomorphically isometric to (CP^m, g_0).*

Proof. Let $(\bar{M}, \bar{g}) := (CP^m, g_0)$. By the Lemma, we have $a_i = \bar{a}_i$, $i = 0, 1, 2$. On the other hand, by a direct computation, we have

$$|B_0|^2 = |B|^2 + \frac{4(m-2)}{m(m+2)}\left(|\rho|^2 - \frac{s^2}{2m}\right). \tag{13}$$

By assumption, (13) implies that $|\rho|^2 = \frac{s^2}{2m}$. Since $(\bar{M}, \bar{g})$ is of constant holomorphic sectional curvature, (3) and the above yield

$$\int_M 2|B_0|^2 \mu_M + \frac{5m^2 + 4m + 3}{m(m+1)}\left[\int_M s^2 \mu_M - \int_{\bar{M}} \bar{s}^2 \mu_{\bar{M}}\right] = 0. \tag{14}$$

Thus, combining (11) and (14) we obtain $B_0 = 0$, $s = \bar{s}$, which completes the proof.

REMARK. From the formula (13), we immediately see that two curvature tensors B_0 and B on a Kähler manifold (M, g) of complex dimension $m \geq 3$ coincides with each other if and only if (M, g) is a Kähler-Einstein manifold.

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