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A BERGMAN–CARLESON MEASURE CHARACTERIZATION OF BLOCH FUNCTIONS IN THE UNIT BALL OF $\mathbb{C}^n$

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1. Introduction

Let B denote the open unit ball of $\mathbb{C}^n$ (throughout this paper n is a fixed positive integer) with its boundary S and ν the Lebesgue measure on B, normalized so that $\nu(B) = 1$. For a function $f \in H^1(B)$, the Hardy space, we say that $f \in BMOA(B)$ if its radial limit function f^* on S is a function of bounded mean oscillations with respect to nonisotropic balls generated by the nonisotropic metric $(\zeta, \eta) \mapsto |1 - \langle \zeta, \eta \rangle|^{1/2}$ on S. See [CRW] for details. A function f holomorphic on B is said to be a Bloch function, or $f \in \mathcal{B}(B)$ if and only if

$$\sup_{z \in B}(1 - |z|^2)|\nabla f(z)| < \infty,$$

where $\nabla f(z) = (\partial f/\partial z_1, \cdots, \partial f/\partial z_n)$ is the complex gradient of f. See [T] for various other characterizations of $\mathcal{B}(B)$. For $\eta \in S$ and $\delta > 0$, we set

$$Q_\delta(\eta) = \{z \in B \ : |1 - \langle z, \eta \rangle| < \delta\}.$$

Our starting point of this research is the following characterization of the space BMOA(B) in terms of Carleson measures, which is well-known (see [G, page 240]) on the discs and has been recently extended to the balls in [CC] : *A holomorphic function f on B belongs to $BMOA(B)$ if and only if*

$$\int_{Q_\delta(\eta)} |\nabla f|^2 - |Rf|^2 d\nu = O(\delta^n)$$

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uniformly in $\eta \in S$ *and* $\delta > 0$.

Here and elsewhere, for holomorphic functions f on $B, \mathcal{R}f = \langle \nabla f, \bar{z} \rangle$ is the radial derivative of f.

It is well-known (see, for example, [CRW]) that the Bloch space $\mathcal{B}(B)$ can be considered as an area version of the space $BMOA(B)$. Motivated by this fact, we prove in this paper the following new characterization of $\mathcal{B}(B)$. In what follows, a positive measure μ on B is called a Bergman-Carleson measure if

$$\mu(Q_\delta(\eta)) = O(\delta^{n+1})$$

uniformly in $\eta \in S$ and $\delta > 0$.

MAIN THEOREM. *Suppose f is a holomorphic function on B. Then $f \in \mathcal{B}(B)$ if and only if $(1-|z|^2)(|\nabla f|^2 - |\mathcal{R}f|)^2 d\nu$ is a Bergman-Carleson measure.*

In the course of proof, we also have some other characterizations of $\mathcal{B}(B)$.

2. Möbius-invariant characterizations of the Bloch space

First we introduce some notations. For $z, w \in \mathbb{C}^n$, let $\langle z, w \rangle = z_1\bar{w}_1 \cdots + z_n\bar{w}_n$ denotes the complex inner product on $\mathbb{C}^n$ and $|z| = \langle z, z \rangle^{1/2}$. For $a, z, \in B, a \neq 0, \varphi_a$ denote the Möbius transformation of B defined by

$$\varphi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2} Q_a z}{1 - \langle z, a \rangle}$$

where $P_a z = \langle z, a \rangle a / |a|^2$ and $Q_a z = z - P_a z$. For $z = 0$, we let $\varphi_0(z) = -z$. The following property of φ_a is very useful in the proof of our main theorem:

$$J_R \varphi_a(z) = \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{n+1} \tag{1}$$

where $z \in B$ and $J_R\varphi_a$ is the real Jacobian of φ_a. See [R, Section 2.2] for details. The invariant Laplacian $\tilde{\Delta}$ defined by

$$\tilde{\Delta} = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \bar{z}_k) \partial^2 / \partial z_j \partial \bar{z}_k$$

where δ_{jk} is the Kroneckers symbol [R].

In the followings, C_α denote a positive constant, depending only on α, which may change on each occasion.

LEMMA 2.1. *Suppose f is a holomorphic function on B with $|\nabla f(z)| \leq M/(1-|z|^2)^\alpha$ for some $\alpha > 0$. If $\eta \in S$ with $\langle z, \eta\rangle = 0$ then*

$$(2) \qquad |\langle \nabla f(z), \bar{\eta}\rangle| \leq \begin{cases} C_\alpha M, & 0 < \alpha < \frac{1}{2}, \\ C_\alpha M \left(1 + \log \dfrac{1}{1-|z|}\right), & \alpha = \frac{1}{2}, \\ C_\alpha M/(1-|z|)^{\alpha - \frac{1}{2}}, & \alpha > \frac{1}{2}, \end{cases}$$

for some positive constant C_α.

Proof. Fix $z \in B$, $z \neq 0$ and write $z = |z|\zeta, \zeta \in S$. Given a point $\eta \in S$ such that $\langle \zeta, \eta\rangle = 0$, we define

$$g(\lambda, \mu) = f(\lambda\zeta + \mu\eta), \quad |\lambda|^2 + |\mu|^2 < 1,$$

then

$$\langle \nabla f(\lambda\zeta), \bar{\eta}\rangle = \frac{\partial g}{\partial \mu}(\lambda, 0)$$

$$(3) \qquad = \langle \nabla f(0), \bar{\eta}\rangle + \int_0^1 \frac{\partial^2 g}{\partial\lambda\partial\mu}(t\lambda, 0)\lambda dt,$$

and

$$(4) \qquad \frac{\partial^2 g}{\partial\lambda\partial\mu}(\lambda, 0) = \frac{1}{2\pi i}\int_{|\mu|=r} \frac{\frac{\partial g}{\partial\lambda}(\lambda,\mu)}{\mu^2} d\mu, \quad |\lambda|^2 + r^2 < 1.$$

Since

$$\frac{\partial g}{\partial \lambda}(\lambda, \mu) = \langle \nabla f(\lambda\zeta + \mu\eta), \bar{\zeta}\rangle,$$

we have from (4) and the hypothesis

$$\begin{aligned} \left|\frac{\partial^2 g}{\partial\lambda\partial\mu}(\lambda, 0)\right| &\leq \frac{1}{r}\sup_{|\mu|=r} |\langle \nabla f(\lambda\zeta + \mu\eta), \bar{\zeta}\rangle| \\ &\leq \frac{M}{r(1-|\lambda\zeta + \mu\eta|^2)^\alpha} \\ &= \frac{M}{r(1-|\lambda|^2 - r^2)^\alpha}. \end{aligned}$$

If we take $r^2 = (1-|\lambda|^2)/2$ then

$$\left|\frac{\partial^2 g}{\partial\lambda\partial\mu}(\lambda,0)\right| \le C_\alpha M/(1-|\lambda|^2)^{\alpha+1/2}.$$

Therefore we have the following estimate from (3)

$$\begin{aligned} |\langle\nabla f(\lambda\zeta),\bar\eta\rangle| &\le |\langle\nabla f(0),\bar\eta\rangle| + \int_0^1 \left|\frac{\partial^2 g}{\partial\lambda\partial\mu}(t\lambda,0)\lambda\right| dt \\ &\le M + C_\alpha M \int_0^{|\lambda|} \frac{dt}{(1-t)^{\alpha+1/2}}. \end{aligned} \tag{5}$$

If we take $\lambda = |z|$ in (5), we have (2) with a constant C_α independent of f.

For a function f holomorphic on B, the complex normal gradient of f is defined as

$$\nabla_N f = \begin{cases} \langle\nabla f, \bar z/|z|\rangle \bar z/|z|, & z \ne 0, \\ 0, & z = 0, \end{cases}$$

and the complex tangential gradient of f is defined as

$$\nabla_T f = \nabla f - \nabla_N f.$$

We note that

$$|\nabla f|^2 = |\nabla_N f|^2 + |\nabla_T f|^2$$

and

$$|\mathcal{R} f(z)| = |z||\nabla_N f(z)|.$$

LEMMA 2.2. *Let f be holomorphic on B and $|\nabla f(z)| \le M/(1-|z|^2)^\alpha$ for some $\alpha > 0$. Then*

$$|\nabla_T f(z)| \le \begin{cases} C_\alpha M, & 0 < \alpha < \frac12, \\ C_\alpha M\left(1+\log\frac{1}{1-|z|}\right), & \alpha = \frac12, \\ C_\alpha M/(1-|z|)^{\alpha-\frac12}, & \alpha > \frac12, \end{cases} \tag{6}$$

for some positive constant C_α.

Proof. Let $z = |z|\zeta$, $\zeta \in S$. If we take orthonormal complements $\eta_2, \cdots, \eta_n$ of ζ, then we have

$$\nabla f(z) = \nabla_N f(z) + \sum_{j=2}^{n} \langle \nabla f(z), \bar{\eta}_j \rangle \bar{\eta}_j.$$

Therefore

$$\nabla_T f(z) = \sum_{j=2}^{n} \langle \nabla f(z), \bar{\eta}_j \rangle \bar{\eta}_j$$

and so

$$|\nabla_T f(z)|^2 = \sum_{j=2}^{n} |\langle \nabla f(z), \bar{\eta}_j \rangle|^2. \tag{7}$$

If we apply Lemma 2.1 to (7), we get (6).

LEMMA 2.3. *If f is holomorphic on B, then*

$$\tilde{\Delta}|f|^2(z) = 4(1 - |z|^2)(|\nabla f(z)|^2 - |\mathcal{R} f(z)|^2).$$

Proof. Easy exercise. See [CC].

The following Möbius-invariant characterizations of the Bloch space $\mathcal{B}(B)$ will play a crucial role in the proof the main theorem.

THEOREM 2.4. *For f holomorphic on B, the followings are equivalent:*

(a) $f \in \mathcal{B}(B)$;
(b) $\tilde{\Delta}|f|^2$ *is bounded in* B;
(c) $\sup_{a \in B} \int_B \tilde{\Delta}|f|^2(z) J_R \varphi_a(z) d\nu(z) < \infty$.

Proof. (a) $\Longrightarrow$ (b) : Suppose $f \in (B)$. Then

$$|\nabla f(z)| \leq M/(1 - |z|^2) \tag{8}$$

for some $M > 0$. By Lemma 2.3,

$$\begin{aligned}
\tilde{\Delta}|f|^2(z) &= 4(1-|z|^2)(|\nabla f(z)|^2 - |\mathcal{R}f(z)|^2)\\
&= 4(1-|z|^2)(|\nabla_N f(z)|^2 + |\nabla_T f(z)|^2 - |z|^2|\nabla_N f(z)|^2)\\
&= 4(1-|z|^2)^2|\nabla_N f(z)|^2 + 4(1-|z|^2)|\nabla_T f(z)|^2\\
&\le 4(1-|z|^2)^2|\nabla f(z)|^2 + 4(1-|z|^2)|\nabla_T f(z)|^2.
\end{aligned}$$

Therefore by (8) and Lemma 2.2, $\tilde{\Delta}|f|^2(z)$ is bounded.

(b) $\Longrightarrow$ (c): It is easy to compute

$$\begin{aligned}
\int_B \tilde{\Delta}|f|^2(z) J_R\varphi_a(z)d\nu(z) &\le \sup_{z\in B}\tilde{\Delta}|f|^2(z)\int_B J_R\varphi_a(z)d\nu(z)\\
&= \sup_{z\in B}\tilde{\Delta}|f|^2(z).
\end{aligned}$$

This shows the implication (b) $\Longrightarrow$ (c).

(c) $\Longrightarrow$ (a) : Let M denote the quantity in (c). Fix $0 < r < 1$. Then by subharmonicity of $|\nabla f|^2 \circ \varphi_a$ we have

$$|\nabla f(a)|^2 \le \frac{1}{r^{2n}}\int_{rB} |\nabla f|^2 \circ \varphi_a(w)d\nu(w).$$

The change of variables $z = \varphi_a(w)$ turns this into

$$|\nabla f(a)|^2 \le \frac{1}{r^{2n}}\int_{\varphi_a(rB)} |\nabla f(z)|^2 J_R\varphi_a(z)d\nu(z).$$

Note that

$$1 - |z|^2 > \frac{(1-r)}{4}(1-|a|^2) \quad \text{for } z \in \varphi_a(rB)$$

and

$$4(1-|z|^2)^2|\nabla f(z)|^2 \le \tilde{\Delta}|f|^2(z) \quad \text{for } z \in B.$$

It follows then that

$$\begin{aligned}|\nabla f(a)|^2 &\le C_r \frac{1}{(1-|a|^2)^2} \int_{\varphi_a(rB)} (1-|z|^2)^2 |\nabla f(z)|^2 J_R\varphi_a(z) d\nu(z) \\ &\le C_r \frac{1}{(1-|a|^2)^2} \int_{\varphi_a(rB)} \tilde{\Delta}|f|^2(z) J_R\varphi_a(z) d\nu(z) \\ &\le C_r \frac{1}{(1-|a|^2)^2} \int_B \tilde{\Delta}|f|^2(z) J_R\varphi_a(z) d\nu(z)\end{aligned}$$

where $C_r = \left(\frac{4}{1-r}\right)^2 / r^{2n}$. Therefore we see that

$$(1-|a|^2)^2 |\nabla f(a)|^2 \le C_r M.$$

This proves the implication (c) $\Longrightarrow$ (a).

3. Proof of Main Theorem

We now give a characterization of Bloch space $\mathcal{B}(B)$ in terms of Bergman-Carleson measures. We begin with the following Möbius-invariant characterization of Bergman-Carleson measures.

PROPOSITION 3.1. *A positive measure μ on B is a Bergman-Carleson measure if and only if*

$$\sup_{a\in B} \int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} d\mu(z) = M < \infty. \tag{9}$$

Proof. Suppose μ is a Bergman-Carleson measure. It is known([CW]) that μ is a Bergman-Carleson measure on B if and only if

$$\int_B |f|^2 d\mu \le C \int_B |f|^2 d\nu \text{ for all } f \in L^2(\nu) \cap H(B), \tag{10}$$

where C is a positive constant independent of f.

Note that

$$J_R\varphi_a = |J_C\varphi_a|^2, \ a \in B,$$

where $J_C\varphi_a$ is the complex Jacobian of φ_a and hence holomorphic on B. Then it follows from (10) that

$$\int_B J_R\varphi_a(z)d\mu(z) \leq C$$

for all $a \in B$. Using the identity (1) we see that

$$\int_B \frac{d\mu(z)}{|1-\langle z,a\rangle|^{2(n+1)}} \leq C(1-|a|^2)^{-(n+1)}$$

for all $a \in B$. Therefore we have (9) with $M = C$.

Suppose (9) holds, and let $Q = Q_\delta(\eta)$ where $\delta > 0$ and $\eta \in S$. Since (9) with $a = 0$ shows that $\mu(B) \leq M$, we can suppose $\delta < 1/2$. Take $a = (1-\delta)\eta$. Then

$$\int_Q \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} \quad d\mu(z) \leq M.$$

On $Q = Q_\delta(\eta)$, we have

$$\begin{aligned} |1-\langle z,a\rangle| &= |1-\langle z,\eta\rangle + \delta\langle z,\eta\rangle| \\ &\leq |1-\langle z,\eta\rangle| + \delta \\ &\leq 2\delta, \end{aligned}$$

and $1-|a|^2 = 1-(1-\delta)^2 = 2\delta-\delta^2$. Therefore

$$\left(\frac{2\delta-\delta^2}{4\delta^2}\right)^{n+1} \int_Q d\mu(z) \leq M,$$

so that

$$\begin{aligned} \mu(Q) &\leq M\left(\frac{2}{1-\delta}\right)^{n+1} \delta^{n+1} \\ &= O(\delta^{n+1}). \end{aligned}$$

Therefore μ is a Bergman-Carleson measure on B.

Proof of Main Theorem. Let f be a holomorphic function on B. By Theorem 2.4, $f \in \mathcal{B}(B)$ if and only if

$$\sup_{a\in B}\int_B \tilde{\Delta}|f|^2(z)J_R\varphi_a(z)d\nu(z) < \infty.$$

Using Lemma 2.3 and the identity (1), we see that

$$\int_B \tilde{\Delta}|f|^2(z)J_R\varphi_a(z)d\nu(z) = 4\int_B \left(\frac{1-|a|^2}{|1-\langle z,a\rangle|^2}\right)^{n+1} d\mu(z),$$

where $d\mu = (1-|z|^2)(|\nabla f|^2 - |\mathcal{R}f|^2)d\nu$. It follows from Proposition 3.1 that $f \in \mathcal{B}(B)$ if and only if μ is a Bergman-Carleson measure on B. The proof is complete.

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