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A SEQUENTIAL APPROACH TO CONDITIONAL WIENER INTEGRALS

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1. Introduction

Let $(C[0,T],\mathcal{F},m_w)$ denote Wiener space where $C[0,T]$ is the Banach space of real valued continuous functions $x(s)$ on the interval $[0,T]$ with $x(0)=0$ under the sup norm. Let F be a real-valued Wiener integrable function on $C[0,T]$. Let X be a function on $C[0,T]$ defined by $X(x)=x(T)$. The conditional Wiener integral of F given X, written $E[F(x)|X(x)=\xi]$, is defined by any Borel measurable and P_X-integrable function of ξ on $(\mathbf{R},\mathcal{B}(\mathbf{R}),P_X)$ such that for all $A\in\mathcal{B}(\mathbf{R})$,

$$\int_{X^{-1}(A)}F(x)dm_w(x)=\int_A E[F(x)|X(x)=\xi]dP_X(\xi)$$

where $\mathcal{B}(\mathbf{R})$ denotes the Borel σ-algebra of $\mathbf{R}$ and $P_X(A)=m_w\circ X^{-1}(A)$ for $A\in\mathcal{B}(\mathbf{R})$. By the Radon-Nikodym theorem, $E[F(x)|X(x)=\xi]$ is unique up to Borel null sets in $\mathbf{R}$. For more details, see [7].

In [1] R.Cameron introduced the concept of a sequential Wiener integral and then used this concept to study Feynman integrals on Wiener space $C[0,T]$. In [7] J. Yeh introduced the concept of conditional Wiener integral $E[F(x)|X(x)=\xi]$ of a function F on $C[0,T]$ given X and proved the inversion formula for conditional Wiener integral, and then used the formula to derive the Kac-Feynman formula and to evaluate conditional Wiener integrals of several functions on $C[0,T]$.

In this paper, motivated by [1] and [7] we give a sequential definition of conditional Wiener integral and then use this definition to evaluate conditional Wiener integral of several functions on $C[0,T]$. The sequential definition is defined as the limit of a sequence of finite dimensional

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Lebesgue integrals. Thus the evaluation of conditional Wiener integrals involves no integrals in function space [cf,5].

2. A sequential definition of conditional Wiener integral

Let $\tau = \{t_1, t_2, \cdots, t_n\}$ be a partition of $[0,T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$. Let $\xi \in \mathbf{R}$ and $x^\xi(s)$ be a polygonal curve in $C[0,T]$ base on a partition τ and the vector $\vec{u} = (u_1, u_2, \cdots, u_{n-1})$ in $\mathbf{R}^{n-1}$ defined by

$$x^\xi(s) = x(s,\tau,\vec{u}) = u_{k-1} + \frac{u_k - u_{k-1}}{t_k - t_{k-1}}(s - t_{k-1}) \tag{2.1}$$

when $t_{k-1} \le s \le t_k$, $k = 1, 2, \cdots, n$, $u_0 = 0$ and $u_n = \xi$.

Let $K^\xi(\tau, \vec{u})$ be the function on $\mathbf{R}^{n-1}$ based on τ defined by

$$K^\xi(\tau,\vec{u}) = \left(\prod_{k=1}^{n} 2\pi(t_k - t_{k-1})\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\sum_{k=1}^{n} \frac{(u_k - u_{k-1})^2}{t_k - t_{k-1}}\right\}$$

where $u_0 = 0, u_n = \xi$. Define $Y_\tau : C[0,T] \to \mathbf{R}^{n-1}$ by $Y_\tau(x) = (x(t_1), x(t_2), \cdots, x(t_{n-1}))$. Let $B \in \mathcal{B}(\mathbf{R}^{n-1})$. Then we have, by the definition of conditional Wiener integral,

$$\int_{X^{-1}(A)} I_{Y_\tau^{-1}(B)} dm_w(x) = \int_A E[I_{Y_\tau^{-1}(B)} | X = \xi] dP_X(\xi) \tag{2.2}$$

for all $A \in \mathcal{B}(\mathbf{R})$, where I_E denotes the indicator function of a subset E of $C[0,T]$. Thus the left hand side of (2.2) is equal to

$$m_w(\{x \in C[0,T] | (Y_\tau(x), X(x)) \in B \times A\}) = \int_A \int_B K^\xi(\tau,\vec{u}) d\vec{u} d\xi.$$

Since $dP_X(\xi) = (2\pi T)^{-\frac{1}{2}} \exp\{-\frac{\xi^2}{2T}\} d\xi$, by using the Radon-Nikodym theorem in (2.2), it follows that

$$E[I_{Y_\tau^{-1}(B)}(x) | X(x) = \xi] = \sqrt{2\pi T} \exp\left\{\frac{\xi^2}{2T}\right\} \int_B K^\xi(\tau,\vec{u}) d\vec{u},$$

almost everywhere ξ in $\mathbf{R}$.

For each $\xi \in \mathbf{R}$, let $C^{\xi}[0,T] = \{x \in C[0,T] | x(T) = \xi\}$. A subset V of $C^{\xi}[0,T]$ of the form

$$(2.3) \quad V = \{x \in C^{\xi}[0,T] | \quad Y_{\tau}(x) \in B\} = Y_{\tau}^{-1}(B), \quad B \in \mathcal{B}(\mathbf{R}^{n-1})$$

is called a cylinder set in $C^{\xi}[0,T]$. Let $\mathcal{R}^{\xi}$ be the collection of all such cylinder sets in $C^{\xi}[0,T]$. Then $\mathcal{R}^{\xi} = \mathcal{R} \bigcap C^{\xi}[0,T]$ where $\mathcal{R}$ is an algebra of cylinder sets in $C[0,T]$, and also $\sigma(\mathcal{R}^{\xi}) = \mathcal{B}^{\xi} \equiv \mathcal{B}(C^{\xi}[0,T]) = \mathcal{B}(C[0,T]) \bigcap C^{\xi}[0,T]$. Define a set function m^{ξ} on $\mathcal{R}^{\xi}$ by

(2.4)

$$m^{\xi}(V) = \sqrt{2\pi T} \exp\left\{\frac{\xi^2}{2T}\right\} \int_B \left(\prod_{j=1}^{n} 2\pi(t_j - t_{j-1})\right)^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} du_1 \cdots du_{n-1},$$

where V is as in (2.3), $u_0 = 0$ and $u_n = \xi$. Then m^{ξ} is a probability measure on $\mathcal{R}^{\xi}$ and so by the Carathéodory extension theorem m^{ξ} has an extension, still denoted by m^{ξ}, to the σ-algebra $\mathcal{F}^{\xi}$ of Carathéodory measurable subsets of $C^{\xi}[0,T]$ with respect to the outer measure induced by m^{ξ} on $\mathcal{R}^{\xi}$ which in particular contains $\mathcal{B}(C^{\xi}[0,T])$, the Borel σ-algebra of $C^{\xi}[0,T]$. The measure m^{ξ} in $C^{\xi}[0,T]$ is called the conditional Wiener measure with parameter ξ.

THEOREM 2.1.[2]. *Let F be a real valued function on $C[0,T]$. Then*

(i) *If F is $\mathcal{F}^{\xi}$-measurable, then F restricted to $C^{\xi}[0,T]$, $F|_{C^{\xi}[0,T]}$ is $\mathcal{F}^{\xi}$-measurable a.e. ξ in $C[0,T]$.*

(ii) *If F is $\mathcal{B}(C[0,T])$-measurable, then $F|_{C^{\xi}[0,T]}$ is $\mathcal{B}^{\xi}$-measurable for every $\xi \in \mathbf{R}$.*

The following theorem shows that the conditional Wiener integral is indeed the integral in $C^{\xi}[0,T]$ with respect to the measure m^{ξ}.

THEOREM 2.2.[2]. *Let F be a real-valued Wiener integrable function on $C[0,T]$ and $X(x)=x(T)$. Then*

(i) $\int_{C[0,T]} F(x)dm(x) = \int_{\mathbf{R}} \int_{C^\xi[0,T]} F(x)dm^\xi(x)dP_X(\xi)$

(ii) *There exists a version of $E[F(x)|X(x)=\xi]$ such that*

$$E[F(x)|X(x)=\xi] = \int_{C^\xi[0,T]} F(x)dm^\xi(x) \tag{2.5}$$

for every $\xi \in \mathbf{R}$.

DEFINITION 2.1. *Let $F(x)$ be a real-valued Wiener integrable function on $C[0,T]$. If the integral in the right hand side of (2.6) exists for all n and if the following limit exists and is independent of the choice of the sequence $\{\tau_n\}$ of partitions such that norm $\|\tau_n\| \to 0$, we say that the conditional Wiener integral with parameter ξ, written $E^s[F(x)|X(x)=\xi]$, exists and is given by*

$$E^s[F(x)|X(x)=\xi] = \lim_{n\to\infty} A_\xi \int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^\xi(\tau_n,\vec{u})d\vec{u} \tag{2.6}$$

where $A_\xi = \sqrt{2\pi T}\exp\left\{\frac{\xi^2}{2T}\right\}$.

THEOREM 2.3. *Let $F(x)$ be a real valued continuous function and $X(x)=x(T)$. If there exists an $R(x) \in L^1(C[0,T])$ such that $|F(x)| \le R(x)$ on $C[0,T]$, then the conditional Wiener integral $E^s[F(x)|X(x)=\xi]$ exists for all parameter ξ and*

$$\begin{aligned} E[F(x)|X(x)=\xi] &= E^s[F(x)|X(x)=\xi] \\ &= \lim_{n\to\infty} A_\xi \int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^\xi(\tau_n,\vec{u})d\vec{u}. \end{aligned}$$

Proof. By the continuity of $F(x)$ on $C[0,T]$ and of x on $[0,T]$, we have

(2.7)

$$\lim_{n\to\infty} F(x((\cdot),\tau_n,Y_{\tau_n}(x))) = F(x), \quad |F(x((\cdot),\tau_n,Y_{\tau_n}(x)))| \le R(x)$$

for all $x \in C[0,T]$. By means of (2.4), we have

$$dm^{\xi} \circ Y_{\tau_n}^{-1}(\vec{u}) = A_{\xi} \cdot K^{\xi}(\tau_n, \vec{u}) d\vec{u}.$$

Hence by change of variable formula [3,p.163], we obtain

$$\begin{aligned} &\int_{C^{\xi}[0,t]} F(x((\cdot), \tau_n, Y_{\tau_n}(x))) dm^{\xi}(x) \\ &= A_{\xi} \int_{\mathbf{R}^{n-1}} F(x((\cdot), \tau_n, \vec{u})) \cdot K^{\xi}(\tau_n, \vec{u}) d\vec{u}. \end{aligned} \tag{2.8}$$

Thus by using (2.7) and dominated convergence theorem in (2.8), we have

$$\lim_{n \to \infty} \int_{C^{\xi}[0,T]} F(x((\cdot), \tau_n, Y_{\tau_n}(x))) dm^{\xi}(x) = \int_{C^{\xi}[0,T]} F(x) dm^{\xi}(x).$$

Hence it follows from (2.5) that the theorem is proved.

3. Evaluation of conditional Wiener integrals

In this section we use the sequential definition of Wiener integral introduced in Section 2 to evaluate conditional Wiener integrals of several functions on $C[0,T]$.

The following lemmas are well known integration formulas which will be used several times in this section.

LEMMA 3.1. *Let b be a positive real number. Then*

$$\frac{1}{\sqrt{2\pi b}} \int_{\mathbf{R}} v^n \exp\left\{-\frac{v^2}{2b}\right\} dv = 1, 0, b \tag{3.1}$$

for $n = 0, 1, 2$ respectively.

LEMMA 3.2. *Let $0 \le t_1 < t_2 < t_3 \le T$. Then*

(3.2)

$$\begin{aligned} &\frac{1}{\sqrt{(2\pi)^2(t_2 - t_1)(t_3 - t_2)}} \int_{\mathbf{R}} \exp\left\{-\frac{1}{2}\left(\frac{(u_2 - u_1)^2}{t_2 - t_1} + \frac{(u_3 - u_2)^2}{t_3 - t_2}\right)\right\} du_2 \\ &= \frac{1}{\sqrt{2\pi(t_3 - t_1)}} \exp\left\{-\frac{(u_3 - u_1)^2}{2(t_3 - t_1)}\right\}. \end{aligned}$$

LEMMA 3.3. *Let* $0 < t_1 < t_2$. *Then for any* $u \in \mathbf{R}$,

(3.3)

$$\frac{1}{\sqrt{(2\pi)^2 t_1(t_2-t_1)}} \int_{\mathbf{R}} v \exp\left\{-\frac{v^2}{2t_1} - \frac{(u-v)^2}{2(t_2-t_1)}\right\} dv = \left(\frac{t_1}{t_2}u\right) \frac{1}{\sqrt{2\pi t_2}} \exp\left\{-\frac{u^2}{2t_2}\right\}.$$

Proof. Observe that

$$\frac{v^2}{t_1} + \frac{(u-v)^2}{t_2-t_1} = \frac{t_2}{t_1(t_2-t_1)}\left(v - \frac{t_1}{t_2}u\right)^2 + \frac{u^2}{t_2} \tag{3.4}$$

and that

(3.5)

$$v \exp\left\{-\frac{v^2}{2t_1} - \frac{(u-v)^2}{2(t_2-t_1)}\right\} = \left\{\left(v - \frac{t_1}{t_2}u\right) + \frac{t_1}{t_2}u\right\} \exp\left\{-\frac{t_2}{2t_1(t_2-t_1)}\left(v - \frac{t_1}{t_2}u\right)^2 - \frac{u^2}{2t_2}\right\}.$$

Thus, by integrating (3.5) with respect to v over $\mathbf{R}$ with the help of Lemma 3.1, we establish (3.3) as desired.

LEMMA 3.4. *Let* $0 < t_1 < t_2$. *Then for any* $u \in \mathbf{R}$,

(3.6)

$$\frac{1}{\sqrt{(2\pi)^2 t_1(t_2-t_1)}} \int_{\mathbf{R}} v^2 \exp\left\{-\frac{v^2}{2t_1} - \frac{(u-v)^2}{2(t_2-t_1)}\right\} dv = \left\{\frac{t_1(t_2-t_1)}{t_2} + \left(\frac{t_1}{t_2}u\right)^2\right\} \frac{1}{\sqrt{2\pi t_2}} \exp\left\{-\frac{u^2}{2t_2}\right\}.$$

Proof. By means of (3.4), we have

$$v^2 \exp\left\{-\frac{v^2}{2t_1} - \frac{(u-v)^2}{2(t_2-t_1)}\right\} \tag{3.7}$$
$$= \left\{\left(v - \frac{t_1}{t_2}u\right)^2 + 2\frac{t_1}{t_2}u\left(v - \frac{t_1}{t_2}u\right) + \left(\frac{t_1}{t_2}u\right)^2\right\}$$
$$\cdot \exp\left\{-\frac{t_2}{2t_1(t_2-t_1)}\left(v - \frac{t_1}{t_2}u\right)^2 - \frac{u^2}{2t_2}\right\}.$$

Thus, by integrating (3.7) with respect to v over $\mathbf{R}$ with the help of Lemma 3.1, we establish (3.6) as desired.

THEOREM 3.5. *Let F be the function on $C[0,T]$ defined by $F(x) = \int_0^T x(s)ds$. Then for $\xi \in \mathbf{R}$,*

$$E^s[F(x)|x(T) = \xi] = \frac{T}{2}\xi.$$

Proof. Let $\tau_n : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of [0,T]. Then by means of (2.1), we have

(3.8)

$$F(x((\cdot),\tau_n,\vec{u})) = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \left(u_{k-1} + \frac{u_k - u_{k-1}}{t_k - t_{k-1}}(s - t_{k-1})\right) ds$$
$$= \sum_{k=1}^n \frac{1}{2}(u_k - u_{k-1})(t_k + t_{k-1}) + (u_{k-1}t_k - u_k t_{k-1})$$
$$= \frac{1}{2}\sum_{k=2}^n (t_k - t_{k-2})u_{k-1} + (T - t_{n-1})\xi.$$

Thus by using (3.8), we obtain

$$\int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^{\xi}(\tau_n,\vec{u})d\vec{u}$$
$$= \frac{1}{2}\sum_{k=2}^n \int_{\mathbf{R}^{n-1}} (t_k - t_{k-2})u_{k-1}K^{\xi}(\tau_n,\vec{u})d\vec{u} + (T - t_{n-1})\xi.$$

But by using Lemma 3.3 $(n-k+1)$-times repeatedly, we have

$$\int_{\mathbf{R}^{n-1}}(t_k-t_{k-2})u_{k-1}K^{\xi}(\tau_n,\vec{u})d\vec{u}=(t_k-t_{k-2})\frac{t_{k-1}\xi}{T\sqrt{2\pi T}}\exp\left\{-\frac{\xi^2}{2T}\right\}.$$

Hence we have

$$\begin{aligned}&A_\xi\int_{\mathbf{R}^{n-1}}F(x(s),\tau_n,\vec{u})K^{\xi}(\tau_n,\vec{u})d\vec{u}\\&=\frac{1}{2}\sum_{k=2}^{n}(t_k-t_{k-2})\frac{t_{k-1}}{T}\xi+(T-t_{n-1})\xi=\frac{T}{2}\xi\end{aligned}$$

so that the theorem is proved.

THEOREM 3.6. *Let F be the function on $C[0,T]$ defined by $F(x)=\int_0^T(x(s))^2ds$. Then for $\xi\in\mathbf{R}$,*

$$E^s[F(x)|x(T)=\xi]=\frac{T^2}{6}+\frac{T}{3}\xi^2.$$

Proof. Let $\tau_n: 0=t_0<t_1<\cdots<t_n=T$ be a partition of [0,T]. Then by means of (2.1), we have

(3.9)

$$\begin{aligned}F(x((\cdot),\tau_n,\vec{u}))&=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(u_{k-1}+\frac{u_k-u_{k-1}}{t_k-t_{k-1}}(s-t_{k-1})\right)^2ds\\&=\sum_{k=1}^{n}\left[\frac{(u_k-u_{k-1})^2}{3(t_k-t_{k-1})}(t_k^2+t_{k-1}t_k+t_{k-1}^2)\right.\\&\qquad+\frac{(u_k-u_{k-1})(u_{k+1}t_k-u_kt_{k-1})}{t_k-t_{k-1}}(t_k+t_{k-1})\\&\qquad\left.+\frac{u_{k-1}t_k-u_kt_{k-1}}{t_k-t_{k-1}}\right]\\&=\sum_{k=1}^{n}\frac{1}{3}(t_k-t_{k-1})(u_k^2+u_{k-1}u_k+u_{k-1}^2).\end{aligned}$$

Thus by using (3.9) we have

(3.10)

$$\int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^{\xi}(\tau_n,\vec{u})d\vec{u}$$
$$=\frac{1}{3}\sum_{k=1}^{n}(t_k-t_{k-1})\int_{\mathbf{R}^{n-1}}(u_k^2+u_{k-1}u_k+u_{k-1}^2)K^{\xi}(\tau_n,\vec{u})d\vec{u}.$$

But by using Lemma 3.3 repeatedly, we obtain

(3.11)

$$\alpha_k\equiv A_\xi\int_{\mathbf{R}^{n-1}} u_k^2 K^{\xi}(\tau,\vec{u})d\vec{u}$$
$$=\frac{t_k(t_{k+1}-t_k)}{t_{k+1}}+\left(\frac{t_k}{t_{k+1}}\right)^2\cdot\frac{t_{k+1}(t_{k+2}-t_{k+1})}{t_{k+2}}+\quad\cdots$$
$$+\left(\frac{t_{n-2}}{t_{n-1}}\right)^2\cdot\frac{t_{n-1}(T-t_{n-1})}{T}+\left(\frac{t_k}{T}\right)^2\xi^2$$

and by using Lemma 3.1 once and then using Lemma 3.3 repeatedly, we obtain

$$A_\xi\int_{\mathbf{R}^{n-1}} u_{k-1}u_k K^{\xi}(\tau_n,\vec{u})d\vec{u}=\frac{t_{k-1}}{t_k}\alpha_k \tag{3.12}$$

where α_k is as in (3.11). Substituting (3.11) and (3.12) in (3.10), and then simplifying, we obtain

$$A_\xi\int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^{\xi}(\tau_n,\vec{u})d\vec{u}$$
$$=\frac{1}{3}\sum_{k=1}^{n}(t_k-t_{k-1})\left(\alpha_k+\frac{t_{k-1}}{t_k}\alpha_k+\alpha_{k-1}\right)$$
$$=\frac{1}{3}\left[\sum_{k=1}^{n}t_{k-1}(t_k-t_{k-1})+\frac{Tt_{n-1}^2}{T^2}\xi^2+\left(T-\frac{t_{n-1}^2}{T}\right)\xi^2\right].$$

Hence we obtain

$$E^s[F(x)|x(T)=\xi]=\frac{1}{3}\int_0^T s\,ds+\frac{T}{3}\xi^2$$

so that the theorem is proved.

THEOREM 3.7. *Let F be the function on $C[0,T]$ defined by $F(x) = \exp\{\int_0^T x(s)ds\}$. Then for $\xi \in \mathbf{R}$,*

$$E^s[F(x)|x(T) = \xi] = \exp\left\{\frac{T^3}{24} + \frac{T}{2}\xi\right\}.$$

Proof. Let $\tau_n : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$. Then by means of (2.1), we have

$$\begin{aligned}
&F(x((\cdot),\tau_n,\vec{u}))\\
&= \exp\left\{\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(u_{k-1} + \frac{u_k - u_{k-1}}{t_k - t_{k-1}}(s - t_{k-1})\right)ds\right\}\\
&= \exp\left\{\sum_{k=1}^{n}\left[u_{k-1}(t_k - t_{k-1}) + \frac{1}{2}(u_k - u_{k-1})(t_k - t_{k-1})\right]\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
G(\tau_n,\vec{u}) &\equiv \sum_{k=1}^{n}\left\{u_{k-1}(t_k - t_{k-1}) + \frac{1}{2}(u_k - u_{k-1})(t_k - t_{k-1}) - \frac{(u_k - u_{k-1})^2}{2(t_k - t_{k-1})}\right\}\\
&= -\frac{1}{2}\sum_{k=1}^{n}\left\{\frac{(u_k - u_{k-1})^2}{t_k - t_{k-1}} - (t_k - t_{k-1})u_{k-1} - (t_k - t_{k-1})u_k\right\}.
\end{aligned}$$

By using (3.4) $(n-1)$ times repeatedly, we obtain

$$\begin{aligned}
&G(\tau_n,\vec{u})\\
&= -\frac{1}{2}\sum_{k=2}^{n-1}\left\{\frac{t_k}{t_{k-1}(t_k - t_{k-1})}\left(u_{k-1} - \frac{t_{k-1}}{t_k}u_k\right)^2 - (t_k - t_{k-2})u_{k-1}\right\}\\
&\quad - \frac{1}{2}\left[\frac{T}{t_{n-1}(T - t_{n-1})}\left(u_{n-1} - \frac{t_{n-1}}{T}\xi\right)^2 + \frac{\xi^2}{T} - (T - t_{n-1})\xi\right]\\
&= -\frac{1}{2}\sum_{k=2}^{n-1}\left[\frac{t_k}{t_{k-1}(t_k - t_{k-1})} - \left(u_{k-1} - \left(\frac{t_{k-1}}{t_k}u_k + \right.\right.\right.\\
&\qquad\left.\left.\left.\frac{(t_k - t_{k-1})(t_k t_{k-1} - t_1 t_0)}{2t_k}\right)\right)^2 - \frac{(t_k - t_{k-1})(t_k t_{k-1} - t_1 t_0)^2}{4t_k t_{k-1}}\right]
\end{aligned}$$

$$-\frac{1}{2}\left[\frac{T}{t_{n-1}(T-t_{n-1})}\left(u_{n-1}-\left(\frac{t_{n-1}}{T}\xi+\frac{(T-t_{n-1})(t_{n-1}t_{n-2}-t_1t_0)}{2T}\right)\right)^2\right.$$
$$\left.-\frac{\xi}{T}(t_{n-1}t_{n-2}-t_1t_0)-\frac{(T-t_{n-1})(t_{n-1}t_{n-2}-t_1t_0)}{4Tt_{n-1}}+\frac{\xi^2}{T}-(T-t_{n-1})\xi\right].$$

Hence we have

$$\int_{\mathbf{R}^{n-1}} F(x((\cdot),\tau_n,\vec{u}))K^{\xi}(\tau,\vec{u})d\vec{u}$$

(3.13)

$$= \int_{\mathbf{R}^{n-1}}\left\{\prod_{k=1}^{n}2\pi(t_k-t_{k-1})\right\}^{-\frac{1}{2}}\exp\{G(\tau_n,\vec{u})\}d\vec{u}$$
$$= \left(\prod_{k=1}^{n}2\pi(t_k-t_{k-1})\right)^{-\frac{1}{2}}\exp\left\{\sum_{k=2}^{n-1}\frac{(t_k-t_{k-1})t_kt_{k-1}}{8}\right\}\exp\left\{\frac{\xi}{2T}t_{n-1}t_{n-2}\right.$$
$$\left.+\frac{(T-t_{n-1})t_{n-2}}{8T}-\frac{\xi^2}{2T}-\frac{(T-t_{n-1})\xi}{2}\right\}$$
$$\times\int_{\mathbf{R}^{n-2}}\exp\left\{-\frac{1}{2}\sum_{k=2}^{n-1}\frac{t_k}{t_{k-1}(t_k-t_{k-1})}\left(u_{k-1}-\left(\frac{t_{k-1}}{t_k}u_k\right.\right.\right.$$
$$\left.\left.\left.+\frac{(t_k-t_{k-1})t_{k-1}}{2}\right)\right)^2\right\}du_1\cdots du_{n-2}$$
$$\times\int_{\mathbf{R}}\exp\left\{-\frac{T}{2t_{n-1}(T-t_{n-1})}\left(u_{n-1}-\left(\frac{t_{n-1}}{T}\xi\right.\right.\right.$$
$$\left.\left.\left.+\frac{(T-t_{n-1})t_{n-1}t_{n-2}}{2T}\right)\right)^2\right\}du_{n-1}.$$

But the last two integrals of the right hand side of (3.13) equals

$$\left(\prod_{k=2}^{n-1}2\pi\frac{t_{k-1}(t_k-t_{k-1})}{t_k}\right)^{\frac{1}{2}}\left(2\pi\frac{t_{n-1}(T-t_{n-1})}{T}\right)^{\frac{1}{2}}\equiv\alpha_{n-1}.$$

So we have

$$\left(\prod_{k=1}^{n}2\pi(t_k-t_{k-1})\right)^{-\frac{1}{2}}\cdot\alpha_{n-1}=\frac{1}{\sqrt{2\pi T}}.$$

Thus (3.13) is equal to

$$\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{\xi^2}{2T}\right\}\exp\left\{\frac{1}{8}\sum_{k=2}^{n-1}(t_k-t_{k-1})t_kt_{k-1}\right\}$$
$$\cdot\exp\left\{\frac{\xi}{2T}t_{n-1}t_{n-2}+\frac{(T-t_{n-1})t_{n-2}}{8T}-\frac{(T-t_{n-1})\xi}{2}\right\}.$$

Hence we have

$$E^s[F(x)|x(T)=\xi]=\exp\left\{\frac{1}{8}\int_0^T s^2ds+\frac{\xi}{2}T\right\}.$$

so that the theorem is proved.

REMARK 3.8. Let $R(x)=\exp\{\beta\|x\|\}, x\in C[0,T], \beta>0$. Then by Fernique Theorem [4, p.159], $R(x)\in L^1(C[0,T])$. Hence the function F in Theorems 3.5, 3.6 and 3.7 satisfy the condition in Theorem 2.4 with this $R(x)$. So we have $E[F(x)|X(x)=\xi]=E^s[F(x)|X(x)=\xi]$ [cf.5].

The following corollary gives formulas for the conditional Wiener integral when the conditioning function is multivalued.

COROLLARY 3.9. *Let* $0=s_0<s_1<\cdots<s_n=T$. *Then we have*

(3.14)
$$E^s\left[\int_0^T x(s)ds\middle|x(s_1)=\xi_1,\cdots,x(s_n)=\xi_n\right]$$
$$=\sum_{k=1}^n\frac{(s_k-s_{k-1})(\xi_k+\xi_{k-1})}{2}$$

(3.15)
$$E^s\left[\int_0^T (x(s))^2ds\middle|x(s_1)=\xi_1,\cdots,x(s_n)=\xi_n\right]$$
$$=\frac{1}{6}\sum_{k=1}^n(s_k-s_{k-1})[(s_k-s_{k-1})+2(\xi_k^2+\xi_k\xi_{k-1}+\xi_{k-1}^2)]$$

(3.16)

$$E^s\left[\exp\left\{\int_0^T x(s)ds\right\}\Big|x(s_1)=\xi_1,\cdots,x(s_n)=\xi_n\right]$$
$$=\prod_{k=1}^n\left[\exp\left\{\frac{(s_k-s_{k-1})^3}{24}+\frac{(\xi_k-\xi_{k-1})(s_k-s_{k-1})}{2}\right.\right.$$
$$\left.\left.+\xi_{k-1}(s_k-s_{k-1})\right\}\right]$$

where $\xi_0=0$.

Proof. Let $F(x)$ denote the function in Theorems 3.5, 3.6 and 3.7. Then since the Wiener process $\{x(s):0\le s\le T\}$ is independent increments [6], it can be shown that

$$E^s[F(x(\cdot))|x(s_k)=\xi_k]=E^s[F(x(\cdot)+\xi_{k-1})|x(s_k-s_{k-1})=\xi_k-\xi_{k-1}].$$

Thus the left hand side of (3.14), (3.15) and (3.16) equals, respectively

(3.14′)

$$\sum_{k=1}^n E^s\left[\int_0^{s_k-s_{k-1}}(x(s)+\xi_{k-1})ds\Big|x(s_k-s_{k-1})=\xi_k-\xi_{k-1}\right]$$

(3.15′)

$$\sum_{k=1}^n E^s\left[\int_0^{s_k-s_{k-1}}(x(s)+\xi_{k-1})^2ds\Big|x(s_k-s_{k-1})=\xi_k-\xi_{k-1}\right]$$

(3.16′)

$$\prod_{k=1}^n E^s\left[\exp\left\{\int_0^{s_k-s_{k-1}}(x(s)+\xi_{k-1})ds\right\}\Big|x(s_k-s_{k-1})=\xi_k-\xi_{k-1}\right]$$

Hence by using the results in Theorems 3.5, 3.6, 3.7 and Remark 3.8, we obtain the results as desired.

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