Bull. Korean Math. Soc. **29** (1992), No. 2, pp. 315–323

SOME REMARKS ON EXTREMALLY CONVERTIBLE MATRICES

SI-JU KIM

1. Introduction

Most problems involving permanents are considerably more difficult than the corresponding problems for determinants. It would therefore be of interest to find a simple method for conversion of permanents into determinants. In general, it is impossible to find a linear transformation T on $Mat_n(R)$, the set of all $n \times n$ real matrices such that $\operatorname{per} A = \det T(A)$ for all $A \in Mat_n(R)$ [4,5,6]. However, for some subclasses of $Mat_n(R)$, there exist linear transformations on them satisfying such property. An $n \times n$-matrix A is called *convertible* if there exists a $(1,-1)$-matrix H such that $\operatorname{per} A = \det(H \circ A)$ where $H \circ A$ denotes the Hadamard product of H and A. Such a matrix H is called a *converter* of A. For a matrix A, let $\pi(A)$ denote the number of positive entries of A. Gibson [1] showed that a convertible n-square $(0,1)$-matrix A with $\operatorname{per} A > 0$ satisfies $\pi(A) \leq \frac{1}{2}(n^2 + 3n - 2)$. Equality holds iff there exist permutation matrices P and Q such that $A = PT_nQ$ (in this case A is said to be *permutation equivalent* to T_n) where $T_n = [t_{ij}]$ is the n-square $(0,1)$-matrix such that $t_{ij} = 0$ iff $j > i + 1$. The structure of $n \times n$ convertible $(0,1)$-matrices A with $\pi(A) = \frac{1}{2}(n^2 + 3n - 2)$ was completely determined by Kräuter and Seifter [4]. An n-square convertible $(0,1)$-matrix is called *extremal* if replacing any zero entry by a 1 breaks the convertibility. Hwang and Kim [2] investigated properties of nonnegative convertible matrices and some extremally convertible matrices. In fact, U_n[2] is an extremally convertible n-square matrix with $\pi(U_n) = 4(n-1)$. Hwang and Kim [3] showed that there exists an extremally convertible matrix A with $\pi(A) = s$ where s is any integer

Received December 10, 1991.

This work was supported by Korea Research Foundation and TGRC-KOSEF in 1991.

with $4(n-1) \leq s \leq \frac{1}{2}(n^2+3n-2)$. We conjecture that if A is an extremally convertible n-square matrix, then $\pi(A) \geq 4(n-1)$. In this paper, we show that the conjecture is true for $n \leq 5$. Also we show that the cardinality of the set of all $H \circ T_n$ with $\det(H \circ T_n) = 2^{n-1}$ is $2^{2(n-1)}$. Let E_{ij} denote the $n \times n$-matrix all of whose entries are 0 except for the (i,j)-entry which is 1. Let $J_{m\times n}$ be the $m \times n$-matrix all of whose entries are 1. For an $n \times n$-matrix A and for $\alpha, \beta \subseteq \{1, 2, \cdots, n\}$, let $A(\alpha|\beta)$ denote the submatrix obtained from A by deleting rows α and columns β and let $A[\alpha|\beta]$ denote the matrix complementary to $A(\alpha|\beta)$ in A.

2. Extremally convertible Matrices

Let $U_2 = T_2$ and let

$$U_n = \begin{pmatrix} U_{n-1} & \mathbf{b} \\ \mathbf{a} & 1 \end{pmatrix}$$

for $n \geq 3$ where

$$\mathbf{a} = \left(0, \cdots, 0, \frac{1-(-1)^n}{2}, 1\right), \quad \mathbf{b} = \left(0, \cdots, 0, \frac{1+(-1)^n}{2}, 1\right)^T.$$

Let's write $T_{n-1} = [\mathbf{x}_1, \cdots, \mathbf{x}_{n-1}]$ and for $k = 1, 2, \cdots, n-1$, let

$$V_{n,k} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{x}_k & \mathbf{x}_1 & \cdots & \mathbf{x}_{k-1} & \mathbf{x}_k & \mathbf{x}_{k+1} & \cdots & \mathbf{x}_{n-1} \end{pmatrix}.$$

Extremally convertible matrices U_n, $V_{n,k}$ and T_n are fully indecomposable. Now we show that there exist extremally convertible matrices which are also partly decomposable. For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \cdots, n\}$.

THEOREM 2.1. *Let $k \times k$-matrix A and $(n-k) \times (n-k)$-matrix B be extremally convertible such that $\operatorname{per} A \cdot \operatorname{per} B \neq 0$. If A has no rows all of whose entries are 1 or B has no columns all of whose entries are 1, then $C = \begin{pmatrix} A & 0 \\ J_{(n-k)\times k} & B \end{pmatrix}$ is extremally convertible.*

Proof. Trivially C is convertible. Since A and B are extremal, no zero entries in A or B can be replaced by 1 to be convertible. Thus it is sufficient to show that if a zero entry in the zero submatrix of C is replaced by 1, then the matrix is not extremal. Let s, t be any integers such that $1 \le s \le k$, $k+1 \le t \le n$. Let A have no rows all of whose entries are 1. Let $\beta = (1, 2, \cdots, k)$ and let $A_j = C[1, 2, ..., s-1, s+1, ..., k, j|\beta]$ where $j \in \{k+1, \cdots, n\}$. We write $C^* = C \circ H$ for any $(1, -1)$-matrix H. Since A is extremal, $\operatorname{per} A_j > |\det A_j^*|$. Let $Q'_{k,n}$ be the set of all elements of $Q_{k,n}$ which does not contains s. Then

$$\begin{aligned}
|\det C^*(s|t)| &= \sum_{j=k+1}^{n} |\det A_j^*||\det B^*(j-k|t-k)| \\
&< \sum_{j=k+1}^{n} \operatorname{per} A_j \operatorname{per} B(j-k|t-k) \\
&= \sum_{\alpha \in Q'_{k,n}} \operatorname{per}(C(s|t))[\alpha|\beta] \operatorname{per}(C(s|t))(\alpha|\beta) \\
&= \operatorname{per} C(s|t).
\end{aligned}$$

Thus $C(s|t)$ is not convertible and hence $C + E_{st}$ is not convertible. Therefore C is extremal. Similarly if B has no columns all of whose entries are $1, C$ is extremal.

LEMMA 2.2. *Let* $A = [a_{ij}] = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ *be extremally convertible with* $\operatorname{per} A > 0$. *Then* $A_{21} = J_{n_1 \times n_2}$ *and* A_{11}, A_{22} *are extremally convertible.*

Proof. Tivially $A_{21} = J_{n_1 \times n_2}$ and A_{11}, A_{22} are convertible. Suppose that A_{11} is not extremal. Then there are integers s, t with $1 \le s, t \le n_1$ such that $a_{st} = 0$ and $A_{11} + E_{st}$ is convertible. Thus there exists a $(1, -1)$-matrix H_{11} such that $\operatorname{per}(A_{11} + E_{st}) = \det((A_{11} + E_{st}) \circ H_{11})$. Since A is convertible, there exists $(1, -1)$-matrix $H = \begin{pmatrix} H' & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ such that $\operatorname{per} A = \det(A \circ H)$. Let

$$K = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

Then

$$\begin{aligned}|\det((A+E_{st})\circ K)| &= |\det((A_{11}+E_{st})\circ H_{11})||\det(A_{22}\circ H_{22})| \\ &= \operatorname{per}(A_{11}+E_{st})\operatorname{per}A_{22} = \operatorname{per}(A+E_{st})\end{aligned}$$

which is impossible. Thus A_{11} is extremal. Similarly A_{22} is also extremal.

Notice that if a partly decomposable $n \times n$-matrix whose permanent is not zero is extremally convertible, then $n \geq 5$. In particular, if the size of A_{11} is 1 or 2, then the size of A_{22} is no less than 4.

CONJECTURE. *Let A be an extremally convertible $n \times n$-matrix. Then $\pi(A) \geq 4(n-1)$, $n \geq 2$.*

Let A be an extremally convertible $n \times n$-matrix. By Gibson [1], A is permutation equivalent to T_3 if $n = 3$. It is easy to show that every extremally convertible 4×4-matrix is permutation equivalent to one of T_4, U_4, S_4 and W_4 where

$$S_4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad W_4 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Thus the conjecture holds for $n = 2, 3, 4$. However, extremally convertible 4×4-matrix A with $\pi(A) = 4(n-1)$ is not unique (upto permutation equivalent) even though $\operatorname{per} A > 0$. We have U_4 and S_4.

LEMMA 2.3. *If every fully indecomposable, extremally convertible $k \times k$-matrix A satisfies $\pi(A) \geq 4(k-1)$ for all positive integer $k \leq n$, then any extremally convertible $n \times n$-matrix B satisfies $\pi(B) \geq 4(n-1)$.*

Proof. Let a partly decomposable $n \times n$-matrix B be extremally convertible. Then trivially $n \geq 4$. If $\operatorname{per} B = 0$, then without loss of generality, we may assume that the form of B is

$$B = \begin{pmatrix} J_{s\times(s-1)} & \mathbf{0} \\ J_{(n-s)\times(s-1))} & J_{(n-s)\times(n-s+1)} \end{pmatrix}.$$

Then

$$\pi(B) = n^2 - ns + s^2 - s.$$

Since W_4 is the unique matrix which is extremally convertible with zero permanent(upto pemutation equivalent) for $n = 4$, it is easy to show that $\pi(B) \geq 4(n-1)$. If $\operatorname{per} B \neq 0$, we may assume that

$$B = \begin{pmatrix} B_{11} & & & \\ & B_{22} & \mathbf{0} & \\ \mathbf{1} & & \ddots & \\ & & & B_{tt} \end{pmatrix}$$

where B_{ii} is a fully indecomposable $n_i \times n_i$-matrix for all $i = 1, 2, \cdots, t$. We will prove by indcution on n. If $n = 4$, trivially $\pi(B) \geq 4(n-1)$. By lemma 2.2, B_{tt} and

$$B' = \begin{pmatrix} B_{11} & & \\ \mathbf{1} & \ddots & \mathbf{0} \\ & & B_{(t-1)(t-1)} \end{pmatrix}$$

are extremally convertible. Thus by hypothesis $\pi(B') \geq 4(n - n_t - 1)$ and $\pi(B_{tt}) \geq 4(n_t - 1)$. Thus

$$\begin{aligned} \pi(B) &= \pi(B') + \pi(B_{tt}) + n_t(n - n_t) \\ &\geq 4(n - n_t - 1) + 4(n_t - 1) + n_t(n - n_t) \\ &= 4(n-1) + n_t(n - n_t) - 4 \geq 4(n-1). \end{aligned}$$

THEOREM 2.4. *The conjecture holds for $n \leq 5$.*

Proof. Clearly the conjecture is true for $n = 2, 3, 4$. Thus we will prove that the conjecture is true for $n = 5$. Let A be an extremally convertible 5×5-matrix. Suppose that A is fully indecomposable. Then for any $i, j, s_{ij} \geq 4$ where $s_{ij} = \sum_k a_{kj} + \sum_l a_{il} - a_{ij}$ [2]. Without loss of generality, we may assume that $s_{11} = \min_{ij} s_{ij}$. Notice that if A has a row(or column) with only two nonzero entries or $s_{11} \geq 7$, then the conjecture is true [cf.2]. Thus we may assume that every row(and column) of A has at least three nonzero entries and $s_{11} = 5$ or 6. If $s_{11} = 6$, every row of $A(1|1)$ has at least three nonzero entries and hence $\pi(A) \geq 6 + 3 \times 4 > 4(5-1)$. If $s_{11} = 5$, we may assume that A is of the form:

$$A = \left(\begin{array}{c|cccc} 1 & 1 & 1 & 0 & 0 \\ \hline 1 & & & & \\ 1 & & * & & \\ 0 & & & & \\ 1 & & & & \end{array} \right).$$

Then every row(and column) of $A(1|1)$ has at least two nonzero entries. In particular, rows(and columns) 3,4 of $A(1|1)$ have at least three nonzero entries. Now we want to prove that A with $(r_1, r_2, r_3, r_4, r_5) = (s_1, s_2, s_3, s_4, s_5) = (3,3,3,3,3)$ is not extremally convertible. Then every extremally convertible 5×5-matrix A which is fully indecomposable satisfies $\pi(A) \geq 4(5-1)$. Let A be an extremally convertible 5×5-matrix with $(r_1, r_2, r_3, r_4, r_5) = (s_1, s_2, s_3, s_4, s_5) = (3,3,3,3,3)$. Then it is easy to show that A is permutation equivalent to one of these forms:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

B is not convertible since $B[4,5|i,j]$ is nonsingular for $(i,j) = (3,4), (3,5)$ and $(4,5)$. Now we show that C is not convertible. Suppose that C is convertible. Then there exists $(1,-1)$-matrix H such that $\operatorname{per} C = \det(C \circ H)$. We write $C \circ H = C^* = [c^*_{ij}]$. Without loss of generality, we may assume that $c^*_{11} = c^*_{22} = \ldots = c^*_{55} = 1$. Then we have $c^*_{12}c^*_{21} = c^*_{13}c^*_{31} = c^*_{34}c^*_{43} = c^*_{25}c^*_{52} = c^*_{45}c^*_{54} = -1$. Also we have $c^*_{12}c^*_{25}c^*_{31}c^*_{43}c^*_{54} = c^*_{13}c^*_{21}c^*_{34}c^*_{45}c^*_{52} = 1$, which is impossible. Now suppose that A is partly decomposable. Then $\pi(A) \geq 4(5-1)$ comes from lemma 2.3.

We have believed that every extremally convertible $n \times n$-matrix with $\pi(A) = 4(n-1)$ is permutation equivalent to U_n for $n \geq 5$ [1,2].

Unfortunately we have another extremally convertible matrix A for $n = 5$. For example, let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Then A is an extremally convertible matrix with $\pi(A) = 4(5-1)$.

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix}$$

is a converter of A.

Let

$$S_{2m} = \begin{pmatrix} S_{2m-2} & B \\ A & J_{2\times 2} \end{pmatrix}$$

for $m \geq 3$ where

$$A = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

and $B = A^T$. Let

$$H_4 = \begin{pmatrix} 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and let

$$H_{2m} = \begin{pmatrix} H_{2m-2} & J_{(2m-2)\times 2} - 2E_{(2m-3)1} - 2E_{(2m-2)2} \\ J_{2\times(2m-2)} & J^*_{2\times 2} \end{pmatrix}$$

for $m \geq 3$ where

$$J^*_{2\times 2} = \begin{pmatrix} 1 & (-1)^{m+1} \\ (-1)^m & 1 \end{pmatrix}.$$

Then H_{2m} is a converter of S_{2m}. Hence S_{2m} is convertible. At the present time I am not able to prove that S_{2m} is extremal except for $m = 2, 3$. If S_{2m} is extremal, we will have another extremally convertible matrices A with $\pi(A) = 4(n-1)$ in addition to U_n.

3. Combinatorial Properties of T_n

Let $\mathcal{C}_n$ (and $\mathcal{C}_n'$) be the set of all $H \circ T_n$ with $\det(H \circ T_n) = 2^{n-1}$ (and -2^{n-1}) where H is any $(1,-1)$-matrix. It is easy to show that $|\mathcal{C}_n| = |\mathcal{C}_n'|$.

LEMMA 3.1. *Let $H \circ T_n = [c_{ij}] \in \mathcal{C}_n'$. If $A_{n,k} \in \mathcal{C}_n$ or $\mathcal{C}_n'$ is such that all columns except k are same as those of $H \circ T_n$ (column k of $A_{n,k}$ may be equal to that of $H \circ T_n$), $1 \le k \le n$, then*

$$A_{n,k} = \begin{cases} H \circ T_n & \text{if } \det(H \circ T_n) = \det A_{n,k}; \\ B_{n,k} & \text{if } \det(H \circ T_n) = -\det A_{n,k} \end{cases}$$

where $B_{n,k}$ is the matrix obtained from $H \circ T_n$ by changing signs of all nonzero entries of column k.

Proof. We will prove by induction on n. Nothing to prove for $n = 1$. Assume that the result holds for all $m < n$. If $k \ge 3$, then $A_{n,k}(1|1) \in \mathcal{C}_{n-1}$ or $\mathcal{C}_{n-1}'$ and all columns except column $k-1$ of $A_{n,k}(1|1)$ are same as those of $H \circ T_n(1|1)$. By hypothesis, $A_{n,k} = H \circ T_n$ or $A_{n,k} = B_{n,k}$. Thus without loss of generality, we may assume that all columns except column 1 of $A_{n,k}$ are same as those of $H \circ T_n$. Notice that $(H \circ T_n)(1|1) = A_{n,k}(1|1)$ with $\det(H \circ T_n)(1|1) = 2^{n-2}$ or -2^{n-2}. Since $\det(H \circ T_n)(1|2) = 2^{n-2}$ or -2^{n-2}, $A_{n,k}(1|2) \in \mathcal{C}_{n-1}$ or $\mathcal{C}_{n-1}'$. By hypothesis, $a_{i1} = c_{i1}$ or $a_{i1} = -c_{i1}$ for all $i = 2, ..., n$. Let $\det A_{n,k} = \det(H \circ T_n)$. If $a_{i1} = c_{i1}$ for all $i = 2, ..., n$, then $\det A_{n,k}(1|2) = -\det(H \circ T_n)(1|2)$ and hence $c_{11}\det(H \circ T_n)(1|1) - c_{12}\det(H \circ T_n)(1|2) = \det(H \circ T_n) = \det A_{n,k} = a_{11}\det A_{n,k}(1|1) - a_{12}\det A_{n,k}(1|2)$. Thus $(c_{11} - a_{11})\det(H \circ T_n)(1|1) = 2c_{12}\det(H \circ T_n)(1|2)$. From this identity we have $a_{11} = -c_{11}$. Then $\det A_{n,k} = -\det(H \circ T_n)$, which is impossible. Therefore $a_{i1} = c_{i1}$ for all $i = 2, ..., n$ and hence $(H \circ T_n)(1|2) = A_{n,k}(1|2)$. Since $\det(H \circ T_n) = \det A_{n,k}$, $a_{11} = c_{11}$ and $A_{n,k} = H \circ T_n$. Similarly if $\det A_{n,k} = -\det(H \circ T_n)$, then $A_{n,k} = B_{n,k}$.

THEOREM 3.2. $|\mathcal{C}_n| = 2^{2(n-1)}$.

Proof. We will prove by induction on n. There is nothing to prove for $n = 1$. Assume that the result holds for all $m < n$. Let $A = [a_{ij}]$ be any element in $\mathcal{C}_n$. Then $a_{11} = 1$ or -1. If $a_{11} = 1$, then $A(1|1) \in \mathcal{C}_{n-1}$. By lemma 3.1, the first column of A is uniquely determined according to $a_{12} = 1$ or -1. By hypothesis, the number of elements of $\mathcal{C}_n$ whose $(1,1)$-entry is 1 is $2 \times 2^{2(n-2)}$. Similarly, the number of elements of $\mathcal{C}_n$ whose $(1,1)$-entry is -1 is $2 \times 2^{2(n-2)}$. Therefore $|\mathcal{C}_n| = 2 \times 2^{2(n-2)} + 2 \times 2^{2(n-2)} = 2^{2(n-1)}$.

COROLLARY. *The number of converter H of T_n is* $2^{\frac{n^2+n-2}{2}}$.

COROLLARY. $|\{V_{n,k} \circ H | H$ *is a converter of* $V_{n,k}\}| = 2^{2(n-1)}$.

Proof. Let $A \in \{V_{n,k} \circ H | H$ is a converter of $V_{n,k}\}$. Then $A(1|1) \in \mathcal{C}_{n-1}$ or $\mathcal{C}'_{n-1}$. Also $A(1|k)$ can be in $\mathcal{C}_{n-1}$ or $\mathcal{C}'_{n-1}$ by permuting columns of A. By the same method as the proof of theorem 3.2, we have the result.

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DEPARTMENT OF MATHEMATICS EDUCATION, ANDONG UNIVERSITY, 760–749