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Estimating Reorder Points for ARMA Demand with Arbitrary Variable Lead Time

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Abstract

In an inventory control system, the demands over time are often assumed to be independently identically distributed(*i.i.d.*). However, the demands may well be correlated over time in many situations. The estimation of reorder points is not simple for correlated demands with variable lead time. In this paper, a general class of autoregressive and moving average processes is considered for modeling the demands of an inventory item. The first four moments of the lead-time demand(L) are derived and used to approximate the distribution of L. The reorder points at given service level are then estimated by the three approximation methods: normal approximation, Charlier series and Pearson system. Numerical investigation shows that the Pearson system and the Charlier series performs extremely well for various situations whereas the normal approximation shows consistent underestimation and sensitive to the distribution of lead time. The same conclusion can be reached when the parameters are estimated from the sample based on the simulation study.

1. Introduction

Let $\{Y_t,\ t \in N\}$ (where N is the set of natural numbers) be a sequence of random variables, and let T be a stopping variable taking on the positive integer values. Then

$$L = \sum_{t=1}^{T} Y_t \tag{1}$$

is a randomly stopped sum of random variables. In an inventory control problem, let Y_t denote the daily demand(or other unit time demand) at discretized time t, and let T be the stochastic lead time which measures the number of days between placing and receiving an order. L is then the lead-time demand of the inventory item. The knowledge of lead-time demand distribution is required for many important inventory control decisions such as determining the

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reorder quantity(Q) and reorder point(R) when both the daily demand and lead time follow some stochastic process. An optimal solution for R and Q by minimizing the total expected cost of a (Q, R) inventory system is obtained, for example, from the following simultaneous equations by Hadley and Whitin([15] pp.163−167) for the stochastic inventory control system.

$$Q = \sqrt{2D(k + b\tilde{S})/h} \tag{2}$$

$$1 - F_L(R) = \frac{hQ}{bD} \tag{3}$$

where R is the reorder point of the inventory item, $\tilde{S}$ is the expected-lost-sales and $F_L(\cdot)$ is the cumulative distribution function($c.d.f.$) of L. Also, D = expected annual demand, k = cost of placing an order, h = annual inventory carrying cost per unit, and b = unit back-order cost. The distribution of L is required for determining $\tilde{S}$, R and Q in (2) and (3). In fact, the $c.d.f.$ of L, $F_L(\cdot)$ can be expressed by

$$F_L(l) = \sum_{t=1}^{\infty} F_{Lt}(l) P_r(T=t) \tag{4}$$

where $P_r(T=t)$ is the probability function of lead time and $F_{Lt}(\cdot)$ is the conditional $p.d.f.$ of L given $T=t$. It is not simple to derive or estimate the distribution of L. It is noted from (4) that the lead-time demand L involves the uncertainty from both the demand Y_t and the lead time T.

Various approaches are available for estimating the distribution of L. One approach assumes that the components of L, that is, the lead time T and demand Y_t are mutually independent and the successive realizations of demand Y_t are identically independently distributed($i.i.d.$). Under these assumptions, the compound distribution of L may be derived by a theoretical argument from the components with parametric assumptions or by some approximation when the exact form is not available. See, for example, Das[11, 12], Burgin and Norman[9], Bott[6], Tadikamalla[25, 26], Ord and Bagchi[20], Bagchi, Hayya and Ord[3] and Bagchi[5]. The crucial assumption made under this approach is that the realizations of the daily demand Y_t are $i.i.d.$ That is, at any point of time, previous demand has no effect on the probabilities of future demand. However, Y_t may well be correlated over time in many practial situations. The compound distribution approach is generally not suitable for a correlated demand structure.

Another approach for estimating the distribution of L is the curve-fitting procedure based on the moments of L. This is a significant improvement from the traditional treatment in a sense that it requires only the existence of the moments of lead time and daily demand without any parametric assumptions for the components nor the $i.i.d.$ assumption on the demand structure. Kottas and Lau[16], Lau and Zaki[18] suggested the curve fitting procedure but with correlated demand and given skewness and kurtosis of L. Ray[21,22,23] suggested procedure for obtaining the reorder points by the moments generating function technique where the daily demands are correlated and normally distributed. Ray has not suggested criteria specified for measuring the performance of the procedure. Lau and Wang[17], Fotopoulos, Wang and Rao[13] relaxed the normality condition and obtained the reorder points by the Pearson curve-fitting and a probability bound, respectively. An, Fotopoulos and Wang[1] obtained similar result but only for the simple correlated demand structure such as AR(1) and MA(1) processes. The general class of ARMA(Autoregressive and moving average) processes are useful for describing demands of many business and economic situations. See. for example, Box and Tiao[8], Madridakis and Wheelwright [19], Ray[23] and Badinelli[2]. The iterative procedure suggested by Box and Jenkins are well established for model identification, parameter estimation, diagnostic checking and forecasting. This procedure for ARMA modeling demand may involve careful analysis of autocorrelation estimates and residuals. Many commercial softwares such as SCA, RATS, SAS and MINTAB are available for this analysis and the cost and time required for the analysis are trivial amount. It is therefore highly desirable to formulate the lead-time demand distribution of L under this class of demand models.

The contributions of the paper are (i) the lead-time demand of L has been formulated by the general class of the correlated ARMA demand model and stochastic lead time, (ii) the exact first four moments of L are derived based on the general stationary ARMA(p,q) demand processes without parametric assumption on lead time. Also the properties of L and the reorder points are investigated. The distribution of L and specially, the reorder points for a given service level are estimated by the Pearson system, a normal approximation and Charlier series. These results are more general than all the previous work. In addition, the Charlier series is a new approach for estimating the distribution of L. It is easier to use and highly accurate as compared with Pearson system. In this paper, the parameters of general ARMA(p,q) demand processes are assumed to be known. This may rais a question on the applicability of this approach in practice. We have twofold answers to this: (i) A subclass of the general model can be identified, based on the moderate size of demand data by Box-Jenkins procedure, for forecasting the distribution of L and the reorder points. (ii) For a given demand model, the parameters involved are usually unknown and must be estimated before a forecast can be made. We have considered this aspect and demonstrated in Table 2 and Table 3 in which the reorder points are obtained based on the estimated parameters for an AR(1) process. Table 3 shows a comparison of reorder points between the known parameters versus the estimated values based on 10,000 simulation trials. This comparison yields, with negligible sampling error, similar to and consistent with the result for known parameters.

2. Modeling of L for ARMA Demand and Variable Lead Time

2.1 ARMA Demand

It is certainly conceivable that an ARMA demand process can result from dynamic behavior in the marketplace. For example, above-average demand in one period may imply below-average demand in a future period and vice versa when customers' demand derives from their inventory positions. In such a situation, the demand process would be ARMA. As Badinelli[2] suggested, there is another situation in which ARMA demand is likely to occur. This deserves some explication since they result from manufacturing practices that are widespread. Consider the situation where a capacity-constrained production facility places demand on a supplier. The demand pattern seen by the supplier is likely to have ARMA characteristics. Because of the capacity constraint, the demand passed on the supplier will be more complex than seen by the production facility. Capacity constraint and the costs of increasing or decreasing capacity motivate a production-smoothing policy. This policy calls for production rates which vary less frequently than demand rates. The decision to increase(decrease) the production rate is often based on the presence of a large(small) queue. The queue represents the cumulative effect of demand and supply over several recent periods and hence, reflects the recent history of demand and supply. In other words, the production planning policy shows some autocorrelation in the production rate of the production facility and consequently in the demand process seen by supplier. Also the capacity constraints cause the uncertainty in the lead time, which becomes random in nature.

2.2 Modeling of Lead-Time Demand and Moments

Suppose that demand Y_t is generated by a ***stationary*** ARMA(p,q) process

$$Y_t = a + \sum_{i=1}^{p}\phi_i Y_{t-i} + \varepsilon_t + \sum_{j=1}^{q}\theta_j \varepsilon_{t-j}. \tag{5}$$

The stationarity of the demand process means that the probability distribution of the demand changes from period to period as a result of depending of demand in any period to period on the actual demand which occurred in previous periods. This dependency is time-invariant that the demand in a future period is state-dependent but not explicitly time-dependent, and the limiting distribution of the demand has finite moments. The uncertainty of the stationary demand and lead time must be considered for an optimal inventory policy. Otherwise ignorance

of such uncertainty often incurs substantial costs to the inventory control. The ARMA(p,q) demand process given by (5) can be expressed recursively as a moving average form with a linear combination of
i.i.d. sequence $\{\varepsilon_t,\ t \in N\}$

$$Y_t = a_t + \sum_{j=0}^{t-1} \psi_j \varepsilon_{t-j} \tag{6}$$

where $\psi_j (j=0,1,\cdots)$ can be computed from the identity on B

$$(\sum_{j=0}^{\infty} \psi_j B^j)(1 + \sum_{i=1}^{p} \phi_i B^i) = 1 + \sum_{j=1}^{q} \theta_j B_j,$$

and a_t which depends on the previous a_t's and Y_t's, has an autoregressive relation

$$a_t = \sum_{i=1}^{p} \phi_i a_{t-1} + a + \sum_{j=t}^{q} \tilde{\varepsilon}_{t-j},$$

with initial values $a_{-i} = \tilde{Y}_{-i}$ *for* $i = 0,1,\cdots,(p-1)$. Also observed residual errors is defined as $\tilde{\varepsilon}_{-j} = \tilde{Y}_{-j} - \hat{Y}_{-j}$ where $\tilde{Y}_{-j}$ and $\hat{Y}_{-j}$ denote actual value and backward forecasted value of the demand at previous period, respectively. Then the lead-time demand L (1) is given by

$$L = \sum_{t=1}^{T} Y_t = \sum_{t=1}^{T} a_t + \sum_{t=1}^{T} S_t \varepsilon_{T+1-t}. \tag{7}$$

where

$$S_t = \sum_{j=0}^{t-1} \psi_j. \tag{8}$$

Expressing L as a linear combination of the random sums of a_t and the *i.i.d.* sequence $\{\varepsilon_t,\ t \in N\}$ makes it rather straightforward to compute the moments of L and to estimate the distribution of L. Noting that the random error $\varepsilon_t (t=1,2,\cdots)$ is independently identically distributed(*i.i.d.*) as normal distribution $N(0,\ \sigma_\varepsilon^2)$, the mean of L is simply

$$\mu = E[L] = E[\sum_{t=1}^{T} a_t]. \tag{9}$$

Hence the deviation from the mean can be written as

$$L - \mu = \mu_t - \mu + \sum_{t=1}^{T} S_t \varepsilon_{T+1-t} = D_T + \sum_{t=1}^{T} S_t \varepsilon_{T+1-t} \tag{10}$$

where $\mu_t = \sum_{t=1}^{T} a_t$ is the conditional mean of L given T and Dt is a deviation from the unconditional mean of L,

$$D_T = \mu_t - \mu. \tag{11}$$

The variance of L is computed as

$$\sigma^2 = E[(L-\mu)^2] = \sigma^2(\mu_t) + \sigma_\varepsilon^2 E[\sum_{t=1}^{T} S_t^2], \tag{12}$$

where $\sigma^2(\mu_l)$ is the variance of conditional mean of lead-time demand for given lead time. Hence this term would be zero for the case of constant lead time. We can see from (12) that the variance of L is dominated by the variance of lead time unless the mean of lead time and variance of random error are large. It implies that the ignorance of lead time variation can result in severe underestimated reorder points. The third central moment of L is obtained as

$$\mu_3 = E[(L-\mu)^3] = E[D_T^3] + 3\sigma_\varepsilon^2 E[D_T\sum_{t=1}^{T} S_t^2]. \tag{13}$$

The second term of (13) indicates that the distribution of L is not symmetric even if the lead time and the random error both have symmetric distributions. This skewness of L explains the reason that the normal approximation to the distribution of L always yields underestimated reorder points as seen in the numerical investigation of Section 5. The fourth moment of L is given as

$$\mu_4 = E[(L-\mu)^4] = E[D_T^4] + 6\sigma_\varepsilon^2 E[D_T^2\sum_{t=1}^{T} S_t^2] + 3\sigma_\varepsilon^4 E[(\sum_{t=1}^{T} S_t^2)^2]. \tag{14}$$

Although the derivation of the first four moments of L is straightforward, D_T of (11) and the second and third moments of L have interesting implication as we have discussed. Also the definition of (10) makes it possible to derive any high moments of L. The exact moments of L can be regarded as a basis for estimating the distribution of L by procedures such as the Pearson system and the Charlier series. The four moments can be used as a framework for investigating the effect of lead time T and the coefficients of ARMA demand process, (ϕ_i, θ_j).

3. Methods of Estimating Reorder Points

In a (Q,R) inventory system, the total expected annual cost is expressed as

$$TC = k\frac{D}{Q} + (\frac{Q}{2} + R - \mu)h + b\frac{D}{Q}\int_R^\infty (l-R)dF_L(L), \tag{15}$$

where the notations of (15) are defined in (2) and (3). The joint determination of (Q, R), minimizing the total cost (15) requires a computer use for complicated search procedure. As is often the case, the order quantity Q is predetermined from a deterministic model without incurring too severe a penalty for missing mathematical optimum. In this situation, the reorder point is estimated at the given service level which results from either managerial consideration or minimizing TC of (15) for given Q. For a predetermined Q, set $d(TC)/dR=0$ to get the best reorder point(R); the first order condition yields

$$1 - \text{service level} = 1 - F_L(R) = \frac{Qh}{Db}. \tag{16}$$

We see from (15) that the distribution of lead-time demand is required to determine the reorder point at a given service level. This section presents three distributions which approximate the distribution of L. Since the first two moments of mean and variance are often insufficient enough to estimate the distribution, the first four moments of L are used to estimate the approximating distributions to the distribution of L.

3.1 Fitting Pearson Curves

The family of distributions, defined by (17), is known as the Pearson system with probability density function($p.d.f.$), $f(\cdot)$

$$\frac{df(x)}{dx} = \frac{(x-a)f(x)}{b_0+b_1x+b_2x^2} \tag{17}$$

The Pearson system (17) is completely determined by its first four moments. and the reorder point at a service level is estimated from the distribution. Solomon and Stephens[24] have shown that excellent approximate percentile points can be obtained, in particular for the long tail distribution, by fitting Pearson curves when the first four moments of the intractable distribution are exactly known. Bowman and Shenton [7] obtained approximating for estimating the percentile points of the Pearson curves. Fitting the pearson curve is well suited for estimating the distribution of L. See, for example, Kottas and Lau[16], Lau and Wang[17] and An, Fotopoulos and Wang[1]. Letting $q(p)$ be the pth percentile of the distribution of standardized L from the Bowman and Shenton formulas, the estimated reorder point of L by the Pearson system is given by

$$R(p) = \mu + q(p)\sigma, \tag{18}$$

where μ and σ defined by (9) and (12) are the mean and standard deviation of the lead-time demand L, respectively.

3.2 Normal Approximation and Charlier Series

Consider the lead-time demand L defined by (7)

$$L = \sum^{T} Y_t = W_T + \sum^{T} a_t,$$

where $W_T = \sum_{t=1}^{T} S_t \varepsilon_{T+1-t}$. Note that W_T is a random sum of ε_t. Since its simplicity to use, the normal approximation has been popularized in education and practice without careful consideration of its appropriateness. One particular concept that is not clearly stated in inventory control text book is that the distribution must be mear symmetric and short tailed. We choose this approximation as a benchmark to evaluate the performance of other approximating distributions. The reorder point at p% service level estimated by the normal approximation is given by

$$R(p) = \mu + Z(p)\sigma, \tag{19}$$

where $Z(p)$ is the pth percentile of the standard normal distribution.

Experience shows that the normal approximation (19) often underestimates the reorder point in many situations. See, for example, Bagchi, Hayya and Chu[4]. An improvement for the approximation naturally leads to the Charlier series where the *c.d.f.* of L is expressed as a series of normal *c.d.f.*'s. See, for example, Kendal and Stuart([15] pp.166−169). Define the second characteristic function, $g(t)$ as

$$g(t) = \log E[e^{itz}] = \log \int_{-\infty}^{\infty} e^{itz}\, dF(z). \tag{20}$$

If $E[Z^r]$ exists, the Taylor expansion of $g(t)$ on t gives

$$g(t) = \sum_{j=0}^{r} k_j \frac{(it)^j}{j\,!} + o(|t|^r) \tag{21}$$

where k_j is the jth cumulant of Z. Without loss of generality, if Z is assumed to be a st andardized variate, then $k_0=k_1=0$, $k_2=1$, $k_3=\alpha_1$, and $k_4=\beta_2-3$, where α_1 and β_2 are skewness and kurtosis of L, respectively. We obtain from (21)

$$E[e^{itz}] \simeq e^{-t^2/2}\,[1+k^3\,(it)^3/6\; + k^4\,(it)^4/24]. \tag{22}$$

Because(See Cramer[10] p.106)

$$\int_{-\infty}^{\infty} e^{itz}\,\phi^{(r)}\,(z)dz = (-it)^r e^{-t^2/2} \tag{23}$$

where $\phi^{(r)}(\cdot)$ is the rth derivative of the standard normal *p.d.f.*, we can invert (23) to obtain

$$F_z(z) = \Phi(z) - \frac{\alpha_1}{6}\Phi^{(3)}(z) + \frac{\beta_2-3}{24}\Phi^{(4)}(z), \tag{24}$$

where $\Phi^{(r)}(\cdot)$ is the rth derivative of the standard normal *c.d.f.* Let $q^c(p)$ be the pth percentile of $F_z(\cdot)$ according to (24). The estimated reorder point by the Charlier series can be then written as

$$R(p) = \mu + q^c(p)\sigma. \tag{25}$$

4. Numerical Investigation

Although only the first four moments of L are required to estimate the distribution of L for which parametric assumption is really neither necessary for the demand structure nor for the distribution of lead time, some usual assumptions, however, are made for the following numerical investigation for the purpose of performance evaluation. They are: (i) the lead time T is distributed as a geometric with probability $Pr(T=t) = \pi(1-\pi)^{(t-1)}$, $t = 1,2,\cdots$, (ii) the random error of the demand process follows a normal distribution $N(0, 16)$, (iii) the demand Y_t is generated by an ARMA(1,1) process

$$Y_t = a + \phi Y_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1} \tag{26}$$

with moving average parameter θ=-0.7, 0.0 or 0.7 and the autoregressive parameter ϕ=-0.9, -0.5, 0.0, 0.5, or 0.9. Note that the ARMA(1,1) process implies an AR(1) process for θ=0.0 and a MA(1), process for ϕ=0.0. Under these parametric assumptions, the distribution of L, or more specifically the reorder point at a service level, which is either given by managerial consideration or determined by (16) minimizing expected inventory cost, is estimated according to (i) the Pearson system (ii) a normal approximation and (iii) the Charlier series when the first four moments of L are obtained by (9), (12), (13) and (14).

In this study, assuming 95% service level, the reorder points are estimated by the proposed methods according to (18), (19) and (25), respectively. Given these estimated reorder points, the exact percentiles are numerically calculated to evaluate the accuracy of the estimation. In fact, let R be the estimated reorder point of the 95th percentile of L, and then following from (4), the probability coverage of R is given by

$$P_r(L < R) = \sum_{t=1}^{\infty} F_{Lt}(R)Pr(T=t),$$

where F_{Lt} is the conditional $c.d.f.$ of L_t, the lead-time demand for given $T=t$. The performance of the estimation methods is then evaluated by a comparison between the probability coverage and the nominal service level of 0.95. When the random errors are $i.i.d.$ as a normal $N(0, \sigma_\varepsilon^2)$, the conditional lead-time demand for given lead time, L_t, follows a normal distribution with mean μ_t in (10) and the variance of $\sigma_t^2 = \sigma_\varepsilon^2\sum_{i=1}^{t} S_i^2$ where S_i is defined as (8). Hence the probability coverage of R is simply

$$Pr(L < R) = \sum_{t=1}^{\infty}\Phi(\frac{R-\mu_t}{\sigma_t})Pr(T=t), \tag{27}$$

where $\Phi(\cdot)$ is the $c.d.f.$ of a standard normal distribution. Numerical results for the estimated reorder points and the corresponding probability coverages are given in the Table 1.

Table 1. Estimated Reorder Point(R) and Probability Coverage(PC) for ARMA(1,1) Demand with $\theta^{a}=-0.7$ and Normal Error N(0,16)

ϕ^{c}	Method[d]	π^{b} 0.20000		0.10000		0.06667		0.0500	
		R	PC	R	PC	R	PC	R	PC
-0.9	P	257	95.0	534	95.0	810	95.0	1,086	95.0
	N	228	92.9	472	92.9	716	92.9	960	92.9
	C	258	95.0	534	95.0	811	95.0	1,087	95.0
-0.5	P	326	95.0	676	95.0	1,026	95.0	1,376	95.0
	N	289	92.9	598	92.9	907	92.9	1,216	92.9
	C	326	95.0	677	95.0	1,027	95.0	1,377	95.0
0.0	P	488	95.0	1,014	95.0	1,539	95.0	2,063	95.0
	N	434	92.9	897	92.9	1,360	92.9	1,823	92.9
	C	489	95.0	1,015	95.0	1,540	95.0	2,065	95.0
0.5	P	975	95.0	2,027	95.0	3,077	95.0	4,126	95.0
	N	867	92.9	1,794	92.9	2,721	92.9	3,647	92.9
	C	978	95.0	2,029	95.0	3,080	95.0	4,129	95.0
0.9	P	4,868	94.9	10,126	95.0	15,375	95.0	20,622	95.0
	N	4,327	93.0	8,966	92.9	13,599	92.9	18,229	92.9
	C	4,881	95.0	10,140	95.0	15,392	95.0	20,642	95.0

[a] : Moving average parameter of ARMA(1,1) demand.
[b] : Parameter of geometric lead time, $Pr(T=t)=(1-\pi)^{t-1}\pi,\ t\geq 1.$
[c] : Autoregressive parameter of ARMA(1,1) demand.
[d] : P=Pearson system, N=Normal approximation, C=Charlier series.

Table 1 shows the reorder points estimated by the Pearson system(P), the normal approximation(N) and the Charlier series(C) for a normal random error N(0,16) with a geometric lead time when the demand follows an ARMA(1,1) process with θ=-0.7. The PC represents the probability coverage of R. Observe that the Pearson system consistently provides nearly perfect estimate for 95% reorder points, where the probability coverages range between 0.949 and 0.950 for various combinations of autoregressive parameters and the lead time distributions. The normal approximation, however, tends to underestimate reorder points in particular for a skewed distribution such as the geometric lead time in Table 1. Experience shows that if the lead time close to a symmetrical form, such as the Poisson distribution with mean larger than 10, even though the distribution of L is skewed, the normal approximation performs better. In fact, referring to (13), we can see that when T is symmetric, $E[D\bar{T}^{3}]\simeq 0$ and thus the skewness of L are reduced as a result. Consequently the normal approximation shows improvement. The Charlier series also yields perfect estimates for the 95% reorder points in this case. Note that the Charlier series can be used to approximate any distribution whereas the Pearson system presumes that the distribution of L belongs to the Pearson family. The Charlier series enables us to explicitly express the *c.d.f.* of L and it provides a more straightforward procedure for estimating reorder points.

Note that the autoregressive parameter, ϕ, has more impact on R than the moving average parameter θ for the ARMA(1,1) model. The difference in R between θ=-0.7 and 0.7(result is not reported here) are so small to the extent that the value of θ in the ARMA(1,1) process could be ignored. This result can be expected since the mean of L, μ, is not dependent on random error and the variance of L is dominated mostly by the variance of lead time(T) and little by the random errors. Also the autoregressive parameter ϕ has a positive effect on R, that is, R increases sharply as ϕ approaches 1. The term of $1/(1-\phi)$ included in the mean of L and (10) makes the reorder point very sensitive to the autoregressive parameter.

5. Implementation and Evaluation

Previous sections on formulating the lead-time demand and estimating the reorder points are based on the assumption that the demand process has been properly identified and the parameters involved are known to us. This situation may not be practical for decision making and the key to resolve this problem lies on the availability and making use of the daily demand data. Assuming that the demand model has been properly identified, say by the Box and Jenkins procedure, two issues are worthwhile to be investigated in order to implement the procedure on estimating the reorder points when the parameters are estimated from the sample data: (i) Is the sampling variation serious? (ii) How large is the approximation error and how large is the sampling errors for various methods? If the approximation and sampling errors are large, then the application of the procedures suggested in this paper is questionable.

To investigate (i) above, a simulation study is performed and the result is summarized below by Table 2. Assuming that an AR(1) demand process with normal error and a geometric lead time are appropriate, two hundred simulation trials are performed. Note that for each simulation trial, the parameters of AR(1) process and the moments of random errors are estimated based on a simulated sample of 120 daily demand; the parameter of the geometric lead time is estimated from a simulated sample of 30 lead times and thus, the estimated probability generating function and moments of T are obtained. Hence the estimated four moments of lead-time demand are computed from (9), (12), (13) and (14). Accordingly the reorder points are estimated by the Pearson system(P), the normal approximation(N) and the Charlie series(C).

Repeating the procedure independently 200 times yields the average reorder point(R), and the corresponding Probability coverage of R and the standard error of R(SE) are listed in the Table 2. As a benchmark for comparison, the simulated value of R by the Monte Carlo(M) are also obtained based on the simulated demand data and lead times.

Table 2. Estimated Reorder Point(R), Probability Coverage(PC) and Standard Error(SE) for AR(1) Demand with Normal Error N(0,16) When Parameters are Estimated*

ϕ[b]	Method[c]	π[a] 0.20000 R	PC	SE[d]	0.10000 R	PC	SE[d]	0.06667 R	PC	SE[d]	0.05000 R	PC	SE[d]
-0.9	P	256	94.9	3.1	530	94.9	6.8	793	94.7	9.2	1,089	95.1	13.8
	N	234	93.4	2.8	484	93.4	6.2	724	93.1	8.4	994	93.6	12.6
	C	264	95.4	3.2	547	95.4	7.0	818	95.2	9.5	1,124	95.5	14.3
	M	252	94.7	4.2	518	94.5	9.6	789	94.6	13.5	1,088	95.0	19.6
-0.5	P	322	94.8	4.0	669	94.8	8.1	1,020	94.9	12.8	1,359	94.8	17.5
	N	292	93.2	3.6	605	93.1	7.3	923	93.2	11.5	1,230	93.1	15.8
	C	330	95.2	4.1	684	95.2	8.3	1,043	95.3	13.1	1,391	95.2	17.9
	M	320	94.7	6.3	667	94.8	11.6	1,037	95.2	18.4	1,341	94.6	24.0
0.0	P	487	94.9	6.4	1,002	94.8	13.3	1,517	94.8	19.3	2,078	95.1	25.7
	N	437	93.1	5.7	898	92.9	11.9	1,358	92.9	17.3	1,859	93.3	23.0
	C	493	95.1	6.5	1,014	95.0	13.5	1,536	95.0	19.6	2,104	95.3	26.0
	M	487	94.9	9.6	1,005	94.9	19.3	1,493	94.5	24.1	2,036	94.8	34.8
0.5	P	982	95.1	12.9	2,027	95.0	24.8	3,013	94.7	37.9	4,089	94.9	52.4
	N	875	93.1	11.4	1,800	93.0	22.0	2,673	92.6	33.6	3,626	92.8	46.4
	C	987	95.2	12.9	2,036	95.1	24.9	3,025	94.7	38.1	4,105	94.9	52.6
	M	974	95.0	20.2	2,026	95.0	36.3	3,006	94.6	54.3	4,057	94.7	73.8
0.9	P	4,808	94.6	57.9	10,156	95.0	124.6	15,102	94.7	193.4	20,953	95.2	273.1
	N	4,273	92.9	51.1	8,991	93.0	110.0	13,357	92.6	170.7	18,522	93.2	241.1
	C	4,820	94.7	57.9	10,169	95.1	124.7	15,118	94.7	193.5	20,973	95.3	273.3
	M	4,759	94.5	91.6	10,153	95.0	183.4	14,757	94.4	250.5	20,832	95.2	374.4

* : 200 simulation trials are performed. The parameters of AR(1) demand and the moments of error are estimated based on 120 simulated daily demands of an AR(1) process for each simulation. The paramter of geometric lead time is estimated from a lead time sample of size 30. Note that Monte Carlo is performed based on the known parameters.

a : Parameter of geometric lead time, $Pr(T=t)(1-\pi)^{t-1}\pi,\ t\geq 1$.

b : Autoregressive parameter of AR(1) demand.

c : P=Pearson system, N=Normal approximation, C=charlier series, M=Monte Carlo.

d : Standard error of the estimated ROP. R.

Observe from Table 2 that the methods P and C give very similar result in the probability coverage of R and the standard error of R. and all are fairly close to the true probability coverage 0.95. The Monte Carlo(M) gives good point estimates of R comparable to P and C, but shows larger variation than P and C. Notice that the method M is based on known parameters whereas methods P and C are based on the estimated ones. The normal approximation (N), however, shows underestimation for all the cases here.

To further investigate the sampling errors and approximation errors, we considered a particular case of AR(1) demand with normal error and geometric lead time with $\pi=0.1$. Ten thousand simulation trial are performed and the results are summarized by Table 3. Observe that Table 3 shows the comparison between the cases where the parameters are given versus the parameters are estimated. Therefore the approximation error due to various error can be separated from the sampling error or total error=approximation error+sampling error, which may be useful for evaluation purpose.

Table 3. Comparison of Estimation Results Based on Known Parameters and Estimated Parameters for Geometric Lead Time with $\pi=0.1$ and AR(1) Demand

		Known Parameters				Estimated Parameters*				
ϕ[a]	Method[b]	R	PC	SE[c]	Approx.[d]	R	PC	SE	Total[e]	Sampling[f]
-0.9	P	533.4	95.00		0	530.1	94.91	0.8	-9	-9
	N	472.0	92.90		-210	483.9	93.37	0.7	-163	47
	C	533.8	95.01		1	546.3	95.36	0.8	36	35
	M**	531.4	94.94	1.0	-6					
0.5	P	2026.0	95.00		0	2017.7	94.93	3.0	-7	-7
	N	1793.1	92.90		-210	1791.7	92.88	2.6	-212	-2
	C	2028.0	95.01		1	2026.0	95.00	3.0	0	-1
	M	2020.4	94.95	3.8	-5					
0.9	P	10126.0	94.99		-1	10099.0	94.95	15.0	-5	-4
	N	8963.4	92.90		-210	8940.8	92.85	13.2	-215	-5
	C	10137.6	95.01		1	10111.6	94.97	15.0	-3	-4
	M	10116.0	94.98	19.4	-2					

* : 10,000 simulation trials are performed. The parameters of AR(1) and of the moments of error are estimated based on 120 simulated daily demands. The parameter of geometric lead time is estimated from simulated observations.

** : Monte Carlo result is obtained from 10,000 simulation trials with known parameters.

a : True autoregressive parameter of AR(1) demand.

b : P=Pearson system, N=Normal Approximation, C=Charlier series, M=Monte Carlo.

c : Standard Error of the estimated ROP, R.

d : Approximation error x 10^{-4}=(Probability coverage of R based on known parameters)-95%.

e : Total error x 10^{-4}=(Probability coverage of R based on estimated parameters)-95%.

f : Sampling error x 10^{-4}=Total error-Approximation Error.

It is clear from Table 3 that (i) in terms of PC, methods P and C are almost perfect when the parmeters are given; (ii) methods P and C have extremely small sampling errors; (iii) methods P, C and M are very comparable in terms of probability coverage of R; (iv) method N again shows consistent underestimation mainly due to the approximation error; (v) the relative value SE/R, ranging from 0.0145 to 0.00253, are very small and thus the variation in sampling study can be negligible. The above empirical study shows that the effect of parameter estimation is small for the cases considered herein and the methods P and C are highly accurate for estimating the reorder points in a stochastic inventory situation even when parameters are estimated from data.

6. Conclusion

The knowledge of distribution of lead-time demand for a stochastic (Q,R) inventory system is frequently required for many basic inventory decisions. In inventory planning, the assumption of *i.i.d.* demand over consecutive time periods is often made. The demand process may be different from *i.i.d.* process and ARMA process is more suitable to model the demand in practice. When the demand process follows an ARMA process with variable lead time, the derivation of the distribution of lead-time demand is almost impossible. This article provides methods of estimating the reorder point for a general stationary ARMA(p,q) demand process with arbitrary lead time. The ARMA(p,q) demand process is expressed as a weighted linear combination of the *i.i.d.* random errors, which in turn makes it rather straightforward to compute exact expression of the first four moments of L, derived as (9), (12), (13) and (14). These four moments of L enable us to estimate the distribution of L by the Pearson system, a normal approximation and the Charlier series.

Numerical investigation on an ARMA(1,1) demand structure shows that both the Pearson system and the Charlier series provide very good estimate of reorder point(R) consistently for various situations while the normal approximation underestimates R especially for asymmetric lead time distribution. The performance of the normal approximation is more sensitive to the lead time than to the distribution of random errors and demand structure. Simulation study on AR(1) demand process shows the same conclusion as when the parameters are estimated from the sample.

It is worthwhile to note that the R is more influenced by the autorgressive parameter ϕ than the moving average parameter ϕ of the ARMA(1,1) demand process. The reorder points increase very rapidly as the autoregressive parameter approaches nonstationary. This indicate that the mistreating demand as *i.i.d.* results in severe underestimation for positive autocorrelation and in overestimation for negative autocorrelation.

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