⊙ 研究論文

Parametrically Excited Vibrations of Second-Order Nonlinear Systems

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2차 비선형계의 파라메트릭 가진에 의한 진동 특성

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Abstract

This paper describes the vibration characteristic of second-order nonlinear systems subjected to parametric excitation. Emphasis is put on the examination of the hydrodynamic nonlinear damping effect on limiting the response amplitudes of parametric vibration. Since the parametric vibration is described by the Mathieu equation, the Mathieu stability chart is examined in this paper. In addition, the steady-state solutions of the nonlinear Mathieu equation in the first instability region are obtained by using a perturbation technique and are compared with those by a numerical integration method. It is shown that the response amplitudes of parametric vibration are limited even in unstable conditions by hydrodynamic nonlinear damping force. The largest response amplitude of parametric vibration occurs in the first instability region of Mathieu stability chart. The parametric excitation induces the response of a dynamic system to be subharmonic, superharmonic or chaotic according to their dynamic conditions.

1. Introduction

The subject of vibration is very important in engineering and science as was well described by Bishop[1] such that "It is no exaggeration to say that it is unlikely that there is any branch of science in which vibration does not play an important role".

Vibration can be classified in several ways according to a damping effect, the characteristics of excitation sources and so on. These are damped vibration, undamped vibration, free vibration, forced vibration, self-excited vibration, parametrically excited vibration, etc. A classical text book such as Nayfeh and Mook[2] introduces such kinds of

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vibration, problems in much detail. Of the vibrations above, the parametrically excited vibration is investigated in this paper with an emphasis on the role of hydrodynamic nonlinear damping force in limiting the response amplitudes.

The parametric vibration denotes the motion of a system with a time-varying stiffness rather than a constant value. In nature, there are many examples of physical systems exhibiting parametric vibration: a pendulum whose support oscillates at an angle with the vertical(Hsu and Cheng[3]), a turbine blade connected to a whirling shaft (Kellenberger[4], mechanisms on vibration foundations(Thompson and Ashworth[5]), a rotating shaft with a time-varying axial thrust (Namachchivaya[6]) and so on. Bolotin[7], and Nayfeh and Mook[2] show several examples of parametrically excited systems in their comprehensive books and cite many references ralated to parametric vibrations.

In the marine field, there are also many dynamic systems undergoing parametric vibration such as ship rolling motion by following or oblique seas(Roberts[8]), lifted load motion of a crane vessel(Patel and Witz[9]), yaw motion of tension leg platform (TLP) by head seas (Rainey[10]), lateral motion of marine risers and TLP tethers by surface platform heave motion(Hsu[11], and Patel and Park[12]), etc. Figure 1 shows typical examples of physical systems exhibiting potential parametric vibrations. Recently, the problems of combined forcing and parametric excitation have been investigated (Park and Patel[13),14)])

Since the parametric vibration is described by the Mathieu equation, the latter has been also extensively studied in the mathematical field prior to the application study of the

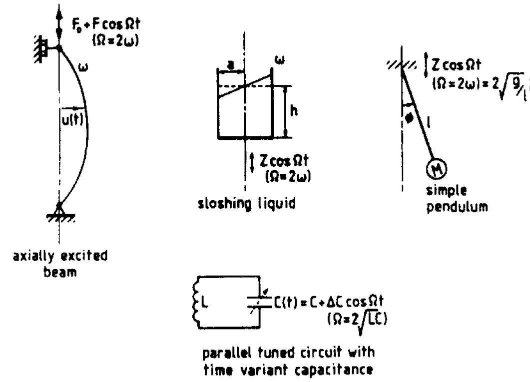

Fig. 1 Examples of physical systems under parametric excitation

parametric vibration. The solutions of the Mathieu equation can be stable or unstable according to a combination of its parameters. This aspect means that the parametric vibration problem involves the nature of dynamic stability. Thus obtaining a stability chart is also important in interpreting the solutions of the Mathieu equation, i.e., parametric vibration problems. The Mathieu equation was first introduced by French scholar E. Mathieu in 1868 when he intended to determine the vibrational modes of a stretched membrane having an elliptical boundary.

Early eminent contributions to the solution of the Mathieu equation are attributed to E. L. Ince and S. Goldstein. Ince[15] computed the characteristic numbers of the Mathieu equation and obtained the stability chart for small parameters. Goldstein[16] seems to be the first systematic contributor to present numerical results for the periodic Mathieu function. McLachlan[17] was also an important contributor to the solution of the Mathieu equation. Most recent research into the Mathieu equation has been carried out to solve the nonlinear equation using perturbation techniques. Using these techniques, linear and nonlinear Mathieu equations have been solved by investigators such as Stoker[18] and Nayfeh[19].

In this study, as a systematic approach to the parametric vibration problem, a canonical form of Mathieu equations is first studied by examining the Mathieu stability chart. Then, the case of nonlinear quadratic damping being added to the canonical Mathieu equation is treated. An analytical solution of the nonlinear Mathieu equation for unstable conditions is obtained by using a perturbation method. However the perturbation method is limited to small magnitude of Mathieu parameters. In order to solve for the case of large magnitude parameters and to confirm analytical results, the fourth-order Runge-Kutta method is used.

2. Theoretical Approach to Mathieu Equation

In this paper, the following system of nonlinear Mathieu equation is considered.

$$\frac{d^2f}{d\tau^2}+(\delta-2q\cos 2\tau)f+c\left|\frac{df}{d\tau}\right|\frac{df}{d\tau}=0 \tag{1}$$

where

$\delta=(2\Omega/\omega)^2$ and $q=(\alpha/2)(2\Omega/\omega)^2$

δ and q in Equation(1) are called the Mathieu parameters. The $2q/\delta(=\alpha)$ in the above equation can be considered to be the strength of parametric excitation. Dynamic systems involving hydrodynamic damping forces, for examples, most marine structures are described by this kind of nonlinear Mathieu equation. Figure 2 shows a typical model of marine structures, where the motions of the surface platform and also the connected vertical slender structure can be described by this nonlinear Mathieu equation.

This nonlinear Mathieu equation poses an interesting but complex problems. Before analysing the nonlinear Mathieu Equation(1), a cannonical form of the Mathieu equation is first studied and then the nonlinear hydrodynamic damping effect is examined. By excluding the nonlinear term in Equation(1), a canonical form of the Mathieu equations can be written as

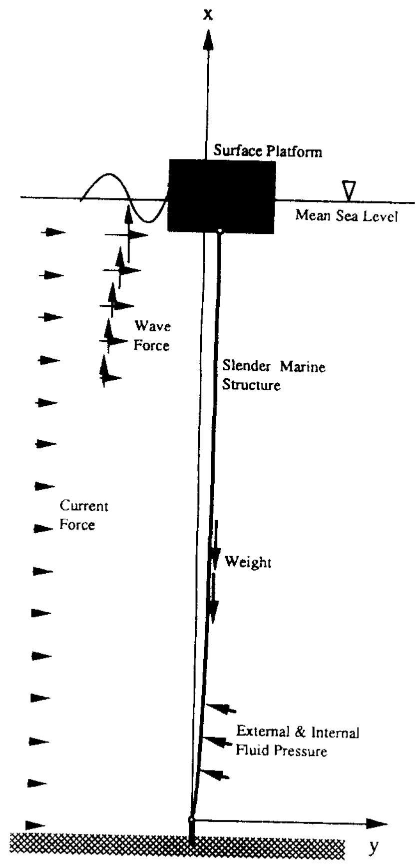

Fig. 2 Typical model of floating marine structures

$$\frac{d^2f}{d\tau^2}+(\delta-2q\cos 2\tau)f=0 \tag{2}$$

A particular characteristic of this Mathieu equation is that it contains a periodically varying coefficient as a special case of the Hill equation. This means that the solutions of the Mathieu equation can be stable or unstable

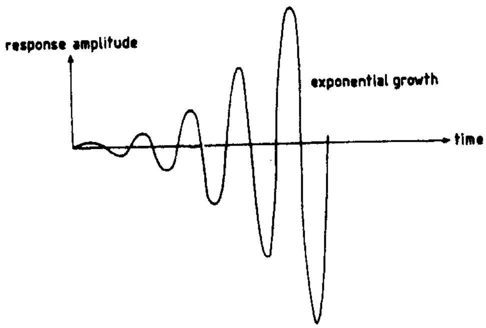

Fig. 3 Response growth of parametric excitation in unstable conditions

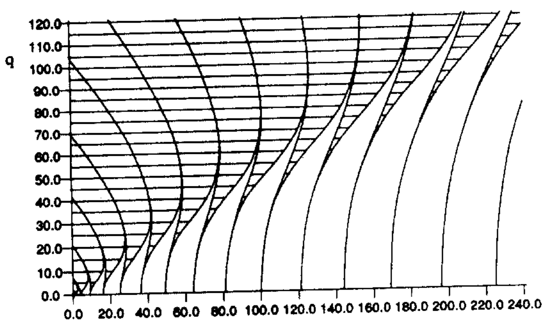

Fig. 4 Mathieu stability chart up to large paramelters
(Shaded regions are unstable)

depending on the combination of value of δ and q. Figure 3 illustrates a response growth of parametric vibrations in unstable conditions. Therefore, one of the main approaches to the Mathieu equation is to obtain a stability chart. The Mathieu stability chart up to large parameters was obtained by Patel and Park[12] and shown in Figure 4.

The shaded regions in Figure 4 indicate unstable conditions where the solutions of the Mathieu equation become exponentially growing as shown in Figure 3. When q approaches to zero, there exist unstable conditions, i.e., $\delta=1(\Omega=\omega/2)$, $\delta=4(\Omega=\omega)$, $\delta=9(\Omega=3\omega/2)$, ……. If q takes larger values, the unstable areas become wider.

In the design of a dynamic system undergoing parametric vibration, it needs to control its dynamic conditions to be away from such unstable conditions.

However, if a nonlinear damping term is included as in Equation(1), even unstable solutions become limited in its amplitude. In order to solve Mathieu type or nonlinear equations, perturbation techniques have been frequently used. There are several methods in this technique such as the straightforward expansion, the method of strained parameters, Whittaker's method, the method of multiple scales and the method of averaging. Of these techniques, only the last two methods are applicable to the nonlinear Mathieu equation. Although such perturbation methods are confined to small parameters, they are very useful in identifying the global pattern of solutions. In this analysis, one of the averaging methods, the Kryov-Bogolieubov-Mitropolsky method is employed(Minorsky[20]).

Assuming that the parameters δ, q, and c in Equation(1) are small and introducing a small parameter ε, Equation(1) can be rewritten as

$$\frac{d^2f}{d\tau^2}+(n^2+\varepsilon\,\underline{\gamma}+\varepsilon\,\underline{q}\,\cos 2\tau)f$$
$$+\varepsilon\,\underline{c}\left|\frac{df}{d\tau}\right|\frac{df}{d\tau}=0 \qquad (3)$$

where,

$$\delta=n^2+\gamma,\ \gamma=\varepsilon\,\underline{\gamma},$$
$$\varepsilon\,\underline{q}=-2q \text{ and } \varepsilon\,\underline{c}=c \qquad (4)$$

According to the Krylov-Bogolieubov-Mitropolsky method, the solution of Equation(3) can be given in the form,

$$f=a\cos\phi+\sum_{n=1}^{N}\varepsilon^n f_n(a,\phi)+O(\varepsilon^{N+1}),$$
$$\phi=n\tau+\theta \qquad (5)$$

where f is a periodic function of ϕ with a period 2π, and a and ϕ are assumed to vary with time according to

$$\frac{da}{d\tau}=\sum_{n=1}^{N}\varepsilon^{n}A_n(a)+O(\varepsilon^{(N+1)}) \quad (6)$$

$$\frac{a\phi}{d\tau}=n+\sum_{n=1}^{N}\varepsilon^{n}B_n(a)+O(\varepsilon^{(N+1)}) \quad (7)$$

Differentiating Equation(5) with respect to t and using Equations(6) and (7) yields

$$\frac{df}{d\tau}=-an\sin\phi+\varepsilon\left(A_1\cos\phi-a\,B_1\sin\phi+\frac{\partial f_1}{\partial\tau}\right) \quad (8)$$

$$\frac{d^2f}{d\tau^2}=-an^2\cos\phi-\varepsilon\left(-2nA_1\sin\phi-2a\,n\,B_1\cos\phi+\frac{\partial^2 f_1}{\partial\tau^2}\right) \quad (9)$$

In order to handle the nonlinear damping term $\left|\frac{df}{d\tau}\right|\frac{df}{d\tau}$ in Equation(3), the following result is useful, that is, if

$$G=G_0+\varepsilon G_1+\varepsilon^2G_2+\cdots \quad (10)$$

then

$$|G|\,G=|G_0|\,G_0+2\varepsilon\,|G_0|\,G_1+O(\varepsilon^2) \quad (11)$$

Thus,

$$\varepsilon\underline{c}\left|\frac{df}{d\tau}\right|\frac{df}{d\tau}=-n^2a^2\varepsilon\underline{c}\,|\sin\phi|\sin\phi \quad (12)$$

Substituting Equations(8), (9) and (12) into Equation(3) and collecting terms of first power in ε yields

$$\frac{d^2f_1}{d\tau^2}+n^2f_1=2nA_1\sin\phi+2anB_1\cos\phi-\underline{\gamma}\,a\cos\phi-a\,\underline{q}\cos2\tau\cos\phi+n^2a^2\underline{c}\,|\sin\phi|\sin\phi \quad (13)$$

In Equation(13), n is an integer and is related to the ordinal of an unstable regions is a Mathieu stability chart. In this analysis, the first region is only considered, that is, $n=1$. Then Equation(13) becomes

$$\frac{d^2f_1}{d\tau^2}+f_1=2A_1\sin\phi+2aB_1\cos\phi-\underline{\gamma}\,a\cos\phi-a\,\underline{q}\cos2\tau\cos\phi+a^2\underline{c}\,|\sin\phi|\sin\phi \quad (14)$$

In order to obtain the solution of the above equation for f_1, further manipulations for the third and fourth terms in Equation(14) are necessary.

By using formulas of trigonometric functions and the Fourier series, the above two terms are respectively written in the following form,

$$\cos2\tau\cos\phi=0.5(\cos\phi+\cos3\phi)\cos2\theta+0.5(\sin\phi+\sin3\phi)\sin2\theta \quad (15)$$

$$|\sin\phi|\sin\phi=\sum_{m=1}^{\infty}b_m\sin m\phi$$

$$m=1,3,5\cdots \quad (16)$$

where, $$b_m=-\frac{8}{\pi m(m^2-4)} \quad (17)$$

Therefore, Equation(14) is rearranged as follows

$$\frac{d^2f_1}{d\tau^2}+f_1=\sin\phi(2A_1+a^2b_1\underline{c}-0.5a\underline{q}\sin2\theta)+\cos\phi(2aB_1-a\underline{\gamma}-0.5a\underline{q}\cos2\theta)+\cos3\phi(-0.5a\underline{q}\cos2\theta)+\sin3\phi(-0.5a\underline{q}\sin2\theta+a^2\underline{c}b_3)+\sum_{m=5}^{\infty}b_m\sin m\phi \quad (18)$$

Removing the secular terms in Equation(18), A_1 and B_1 are determined as follows:

$$A_1=0.25a\underline{q}\sin2\theta-0.5a^2b_1\underline{c} \quad (19)$$

$$B_1=0.5\underline{\gamma}+0.25\underline{q}\cos2\theta \quad (20)$$

Thus Equation(18) becomes in the form

$$\frac{d^2f_1}{d\tau^2}+f_1=-0.5a\underline{q}\cos2\theta\cos3\phi-0.5a\underline{q}\sin2\theta\sin3\phi+a^2\underline{c}b_3\sin3\phi+\sum_{m}^{\infty}b_m\sin m\phi \quad (21)$$

where,

$m=5, 7, 9, \cdots$

The particular solution of Equation(21) takes in the following form

$$f_1 = (aq/16)(\cos 2\theta \cos 3\phi + \sin 2\theta \sin 3\phi + a^2 c \sum_{m}^{\infty} \frac{b_m}{1-m^2} \sin m\phi \quad (22)$$

$m=5, 7, 9, \cdots$

Substituting Equation(22) into Equation(5) to a first approximation and replacing these parameters by the original ones in Equation(4) yields

$$f = a\cos(\tau+\hat{\theta}) - (\frac{aq}{8})\{\cos 2\theta \cos 3(\tau+\theta) + \sin 2\theta \sin 3(\tau+\theta)\} + a^2 c \sum_{m}^{\infty} \frac{b_m}{1-m^2} \sin m(\tau+\theta)$$

$$m=5, 7, 9, \cdots \quad (23)$$

Now, a and θ are to be determined from Equations(6) and (7), respectively, to a first approximate such that

$da/d\tau = \varepsilon A_1$ and $d\phi/d\tau = 1+\varepsilon B_1$

Substituting Equations(19) and (20) into the above equation and using $\phi = \tau + \theta$, yields

$$da/d\tau = -0.5\, aq \sin 2\theta - 0.5\, a^2 b_1 c \quad (24)$$

$$d\phi/d\tau = 0.5\gamma - 0.5 q \cos 2\theta \quad (25)$$

In the nonlinear Mathieu Equation(1), the unbounded solutions are limited by nonlinear damping effects even though the system is in an unstable condition. It is, therefore, important to obtain steady-state periodic solutions with stationary amplitude $\hat{a}$ and phase angle $\hat{\theta}$. These are obtained by setting $da/d\tau=0$ and $d\phi/d\tau=0$ in Equations(24) and (25). The nontrivial solutions of Equations(24) and (25) satisfies

$$-q\sin 2\hat{\theta} - \hat{a} b_1 c = 0 \quad (26)$$

$$\gamma - q\cos 2\hat{\theta} = 0 \quad (27)$$

Eliminating $\hat{\theta}$ from the above two equations and using $b_1 = 8/(3\pi)$ from Equation(17) and $\gamma = \delta - 1$ from Equation(4) gives

$$\hat{a} = \frac{3\pi}{8c}\sqrt{q^2-(\delta-1)^2} \quad (28)$$

From Equations(26) asnd (27), $\hat{\theta}$ also can be given in the form

$$\hat{\theta} = \frac{1}{2}\sin^{-1} - \sqrt{1-\frac{(\delta-1)^2}{q^2}} \quad (29)$$

The above result is related to the first instability region of the Mathieu chart and provides important information for the nonlinear Mathieu equation with a quadratic damping term. As can be seen from Equation(28), the response amplitudes are limited even if a system is in the instability region. Substituting Equation(28) into Equation(23) gives a complete solution of parametric vibrations of the model in the first instability of the Mathieu chart. One important result is the steady-state response amplitude, $\hat{a}$ which is plotted against δ and q in Figure 5. $\hat{a}$ has its maximum value at the centre of the instability region, i. e.,

$$\hat{a}_{max} = \frac{3\pi}{8c} q, \text{ for } \delta = 1 \quad (30)$$

It can be seen that $\hat{a}_{max}$ is proportional to the magnitude of the parametric excitation, q and inversely proportional to the damping coefficient, c.

Equation(30) can be expressed in terms of the natural frequency and excitation freuency from equation(1).

$$\hat{a}_{max} = \frac{3\pi\alpha\Omega^2}{4c\omega^2}, \text{ for } \Omega = \frac{\omega}{2} \quad (31)$$

Equation(31) indicates that maximum

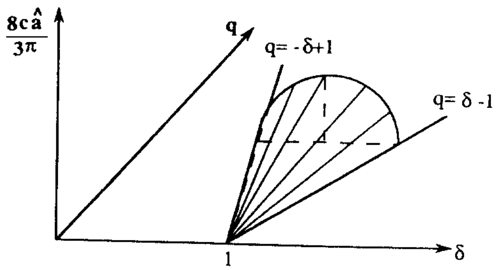

Fig. 5 Stationary amplitude of the nonlinear Mathieu equation in the first instability region

response amplitude occurs when the excitation frequency is twice as large as the natural frequency. It should be noted that the above result is applicable to the first instability region only. However the general response pattern in the higher instability region is the same as the first one. In order to obtain the steady-state solution analytically for higher order instability regions, higher order approximations are needed, which makes the calculation complicated.

3. Numerical Results and Discussions

The nonlinear Mathieu equation with large magnitude parameters cannot be solved analytically. Therefore, it is necessary to employ a numerical method. The previous analytical results of parametric excitation for small magnitude parameters also need to be verified by a numerical method. Among numerical methods for ordinary differential equations, the fourth-order Runge-Kutta method is widely used due to its stabilty and accuracy. In this analysis the Runge-Kutta method is employed to obtain the solution of the nonlinear Mathieu Equation(1) with large parameters.

The following initial conditions are used for numerical solutions.

$$f(0)=0.1,\ df(0)/d\tau=0 \qquad (32)$$

Equation(1) is then solved numerically for different values of parameters, δ and q. To begin with, the frequency response curve of parametric vibrations is obtained as a function of $\delta\ (=[2\Omega]^2/\omega^2)$ parameter. As can be seen from the previous analytical result, equation(30), the response also depends on the value of damping coefficient, c. In Figure 6, the frequency response curve is plotted for four different damping coefficient. The strength of the parametric excitation, $2q/\delta\ (=\alpha)$ is set to be 1.0 for Figure 6. With reference to the Mathieu stability chart in Figure 4, the following conclusions can be drawn from Figure 6 ;

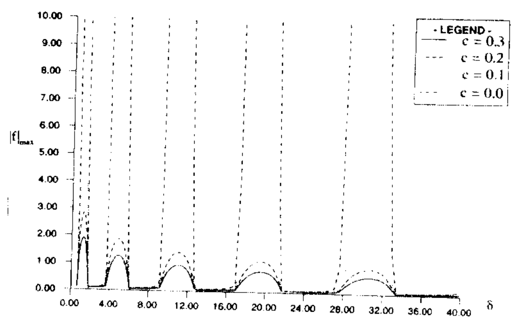

Fig. 6 Response diagram of parametric excitation for different damping coefficients

1) When the damping coefficient is zero, the response amplitude becomes unlimited in every instability region.

2) When the damping effect is considered, even the response amplitude of unstable solutions become limited.

3) The magnitude of limited response amplitde becomes larger as the instability region gets lower.

4) the response amplitude is the largest in the centre of each instability region.

Time histories of parametric vibration are

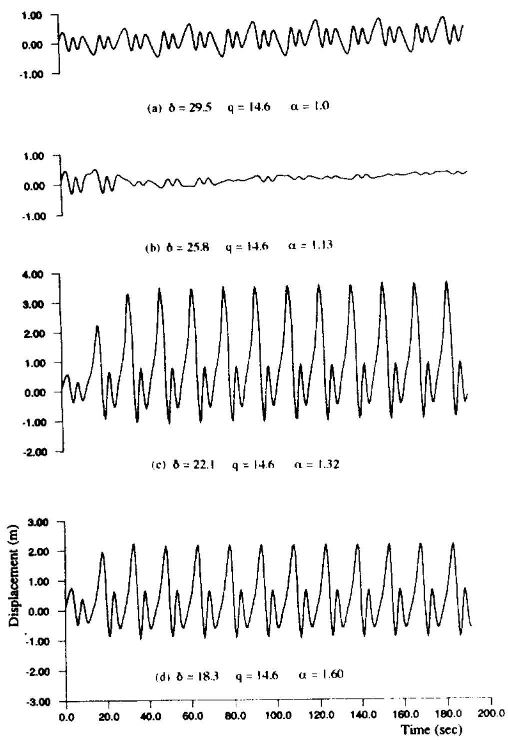

Fig. 7 Time histories of parametric vibration in around the fourth and the llfth instability regions

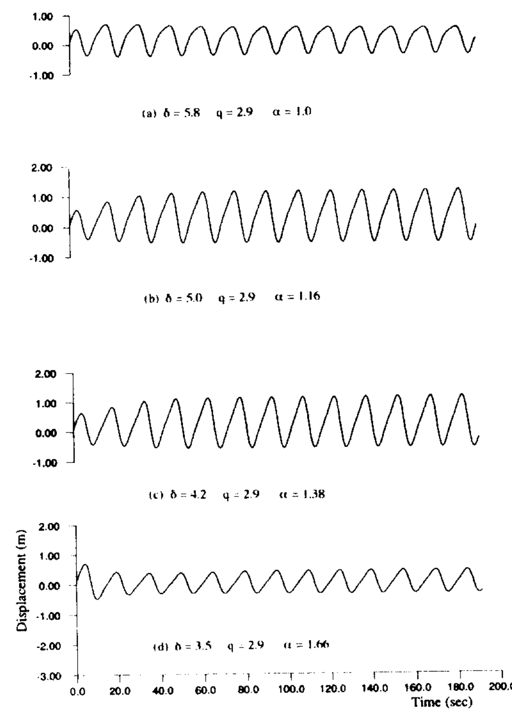

Fig. 8 Time histories of parametric vibration in the second instability region

then obtained for typical dynamic conditions. Some results are illustrated for slender marine structures sudjected to parametric excitation. The excitation period is taken to be 15 seconds. Figure7 illustrates the time histories of the structures whose dynamic conditions correspond to around the fourth and the fifth instability regions of the Mathieu stability chart. Some interesting results can be found by comparing the response amplitude of Figure 7(a) with that of Figure 7(b). Even when the excitation strength is increased, the response amplitude gets smaller which is unexpected. This is the characteristic of parametric vibration as can easily be shown from the stability shart of Figure 4. The reason is that the condition of Figure 7(a) corresponds to the centre of the fifth instability area. Meanwhile the conditions of Figure 7(b) are near the stability region in the chart.

However, as can be seen in Figure 7(c), the response amplitudes sharply increase with the excitation strength due to the dynamic condition appraching the fourth instability region of the Mathieu chart. Further increase in excitation strength (Figure 7(d)) results in the decrease of the response amplitude again because of the dynamic condition approaching stable regions. In addition, the response frequency also depends on the strength of parametric excitation, $2q/\delta\,(=\alpha)$. The oscillation in Figure 7(b) exhibits chaotic motion but those in Figure 7(a), (c) and (d) show perioddoubling phenomena for excitation period of 15 seconds.

Figure 8 shows results for the structures whose dynamic conditions correspond to the second instability region of the Mathieu chart. The trend of amplitude variation with the increase of excitation strength is the same as that for the above mentioned instability region. Even though the excitation strength is increased, the corresponding response amplitudes can be smaller due to the dynamic conditions. The respose frequency is the same as that of excitation frequency regardless of the excitation strength. One of the notable results of the above parametric excitation is that the response frequency of parametric excitation depends on its dynamic conditions. The response frequency of parametric excitation can be period multiplications or chaotic according to the dynamic conditions.

However, approximately speaking, the response frequency of parametric excitation is half of the excitation frequency in the first instability region, equal in the second region and double in the fourth regions and so on. Therefore, in the parametric excitation case, the following frequency relationship can be obtained between the response frequency(ω_r) and the excitation frequency(ω)

$$(\omega_r) = (n/2)\,\omega$$

where n denotes the instability region number. These aspects are contrary to the forced vibration case where the response frequency is usually identical to the excitation freuency in this kind of nonlinear system.

4. Conclusions

This work has been carried out to investigate the parametrically excited vibrations of a system with a nonlinear hydrodynamic damping term. To begin with, the Mathieu problem was described by examining Mathieu stability chart. The response characteristics of the parametric excitation are obtained for the first instability region by analytcally solving the nonlinear Mathieu equation. For higher-order instability regions, the fourth-order Runge-Kutta method is used to obtain the frequency response curve. From these the following properties for parametric vibrations with a nonlinear hydrodynamic damping term can be drawn

(1) Even though dynamic conditions fall under unstable regions, the responses are limited dynamically stable due to the hydrodynamic damping force.

(2) The limited response amplitudes are the largest at the centre of each instability region and come to zero in stable areas or boundary lines between unstable regions.

(3) The response amplitudes of parametric excitation rely on its strength, $2q/\delta$ and the hydrodynamic damping force.

(4) The response frequency of parametric excitation is half of the excitation frequency in the first instability region, equal in the second region and becomes multiple in the higher instability regions. In some cases, there also exists chaotic motion.

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