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A Study on a Polynomial Representation of Finite Posets Using Order Preserving Maps*

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1. Introduction

Let X be a finite poset, and let A be a finite subposet of the chain of natural numbers $\mathrm{N} = \{1, 2, \ldots, n, \ldots\}$. Now, we will use the following notations :

$K_1 = \{f : X \to A \mid f$ is an order preserving map$\}$,
$K_2 = \{f : X \to A \mid f$ is a strictly order preserving map$\}$,
$K_3 = \{f : X \to A \mid f$ is an order preserving surjective map$\}$,
$K_4 = \{f : X \to A \mid f$ is a strictly order preserving surjective map$\}$,
$K_5 = \{f : X \to A \mid f$ is an order preserving injective map$\}$,
$K_6 = \{f : X \to A \mid f$ is an order preserving bijective map$\}$,

A polynomial $p_j(X)$ using an order preserving map is defined by

$$p_j(X) = \sum_{k=1}^{\infty} a_k z^k,$$

where $a_k = |K_j|$, $|A| = k$ and $j = 1, 2, 3, 4, 5, 6$.

Also let x_k be a vertex of a finite poset X, and let i_k be an element of a chain $A_n = \{1, 2, \ldots, n\}$. Then the polynomial defined by

$$F(K_j; X; x_k; i_k; n) = \sum_{n=i}^{\infty} N(K_j; X; x_k; i_k; n)x^n, \quad i \text{ is the maximum of } i_k\text{'s}$$

is said to be a Generalized Stanley-Daykin-Paterson series (GSDP-series, for short) if $N(K_j; X; x_k; i_k; n)$ is the number of order preserving maps $\omega : X \to A_n$ in K_j such that $\omega(x_k) = i_k$ for each k.

It is obvious that $X \cong Y$ implies that $F(K_j; X; x_k; i_k; n) = F(K_j; Y; y_k; i_k; n)$ for all j. But the converse is not true. In this paper we will show the following theorems :

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Theorem 3.12. *Let $\{x_1, \ldots, x_n\}$ be a maximal chain of a P–graph $X = \bar{m}_1 \oplus \cdots \oplus \bar{m}_n$ and let $\{y_1, \ldots, y_n\}$ be a maximal chain of Y with $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$. Suppose that $(X; x_1, \ldots, x_n)$ and $(Y; y_1, \ldots, y_n)$ are GSDP-equivalent in K_j for all $j = 1, \ldots, 6$. Then X and Y are isomorphic.*

Theorem 3.13. *Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be maximal chains of families X and Y with $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ respectively, where $n - 1 = h(X)$. Suppose that $(X; x_1, \ldots, x_n)$ and $(Y; y_1, \ldots, y_n)$ are GSDP-equivivlaent in K_j for all $j = 1, \ldots, 6$. Then X and Y are isomorphic.*

The notations in this paper are standard. They are taken from [7]. Specially, we denote by $\binom{n}{r}$ and ${}_nH_r$ the number of unordered selections without repetition and with repetition, of r objects from n objects, respectively. Also, we denote by ${}_nP_r$ and ${}_n\Pi_r$ the number of ordered selections without repetition and with repetition, of r objects from n objects, respectively. Throughout this paper our poset is finite.

2. Preliminary results

In this section we will discuss some examples of a polynomial $p_j(X)$ and some properties of posets, which will be used later.

Example 2.1. Let C_n be a chain. Then :

(1) $p_1(C_n) = \sum_{k=1}^{\infty} {}_kH_n z^k = \sum_{k=1}^{\infty} \binom{n+k-1}{k-1} z^k.$

(2) $p_2(C_n) = \sum_{k=0}^{\infty} \binom{n+k}{k} z^{k+n} = z^{n-1} p_1(C_n).$

(3) $p_3(C_n) = \sum_{k=1}^{n} {}_kH_{n-k} z^k = \sum_{k=1}^{n} \binom{n-1}{k-1} z^k.$

(4) $p_6(C_n) = p_4(C_n) = z^n.$

(5) $p_5(C_n) = p_2(C_n).$

Example 2.2. Let $\bar{n}$ be an antichain. Then :

(1) $p_1(\bar{n}) = \sum_{k=1}^{\infty} ({}_k\Pi_n) z^k.$

(2) $p_2(\bar{n}) = p_1(\bar{n}).$

(3) $p_3(\bar{n}) = z + (2^n - 2)z^2 + (3^n - 3(2^n - 1))z^3 + \ldots$

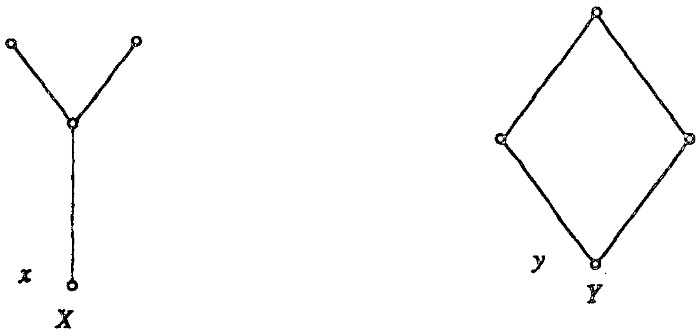
(4) $p_4(\bar{n}) = p_3(\bar{n})$.

(5) $p_5(\bar{n}) = \sum_{k=0}^{n-1} {}_{n+k}P_n z^{n+k}$.

(6) $p_6(\bar{n}) = n!z^n$.

Definition 2.3. A bijective map $f : X \to Y$ is said to be an isomorphism if $x \leq y$ in X if and only if $f(x) \leq f(y)$ in Y. Two posets X and Y are said to be isomorphic if there is an isomorphism $f : X \to Y$, and it is denoted by $X \cong Y$.

Example 2.4. Let X and Y be posets with the following diagrams :



(1) $F(K_1; X; x; i; n+i-1) = \sum_{n=i}^{\infty} \frac{n(n+1)(2n+1)}{6} z^{n+i-1} = F(K_1; Y; y; i; n+i-1)$

(2) $F(K_2; X; x; i; n+i-1) = \sum_{n=i}^{\infty} \frac{(n-2)(n-1)(2n-3)}{6} z^{n+i-1}$
$= F(K_2; Y; y; i; n+i-1)$

(3) $F(K_3; X; x; i; n) = F(K_3; Y; y; i; n) = \begin{cases} z + 4z^2 + 5z^3 + 2z^4 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$

(4) $F(K_4; X; x; i; n) = F(K_4; Y; y; i; n) = \begin{cases} z^3 + 2z^4 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$

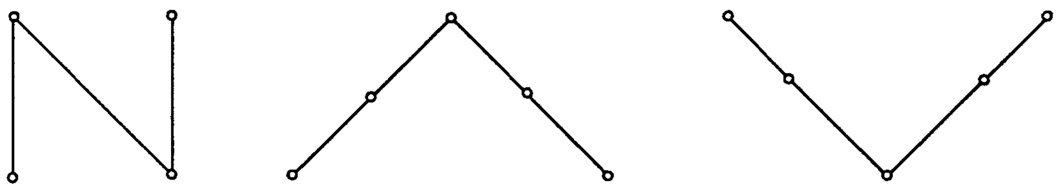
(5) $F(K_5; X; x; i; n+i-1) = \sum_{n=i}^{\infty} \frac{(n-3)(n-2)(n-1)}{3} z^{n+i-1}$
$= F(K_5; Y; y; i; n+i-1)$

(6) $F(K_6; X; x; i; n) = F(K_6; Y; y; i; n) = \begin{cases} 2z^4 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$

Hence X and Y are not isomorphic, but $F(K_j; X; x; i; n) = F(K_j; Y; y; i; n)$ for all i.

Definition 2.5. Let x and y be vertices of a finite poset X. Then x is said to be a lower (or an upper) cover of y if $x < y$ (or $x > y$) and there is no z in X such that $x < z < y$ (or $x > z > y$) and it is denoted by $x \propto y$ (or $y \propto x$).

Definition 2.6. Let N, A and $\forall$ be posets with the following Hasse diagrams (a), (b) and (c) respectively.



(a) (b) (c)

A poset X is said to be P–free if it contains no cover preserving subposets isomorphic to the poset with Hasse diagram P.

Definition 2.7. A poset is said to be series-parallel if it can be decomposed into singletons using ordinal sum and disjoint sum. In particular, a poset X is said to be a P–graph if it can be decomposed into antichains using ordinal sum.

Definition 2.8. A finite poset is series-parallel if and only if it contains no subposet isomorphic to the poset N. That is, every series-parallel poset is an N–free poset.

Proof: It follows from [9].

Proposition 2.9. *Every connected series-parallel poset of height* 1 *is a P–graph.*

Proof: The proof follows from [6].

Definition 2.10. Let x be a vertex of a finite poset X. Then we define the following numerical functions :

$$f_1(x) = |\{y \in X : x < y\}|, \quad \text{the number of descendants of } x,$$
$$f_2(x) = |\{y \in X : x > y\}|, \quad \text{the number of ancestors of } x.$$

A poset X is said to be a family if both $f_1(x) > f_1(y)$ and $f_2(x) < f_2(y)$ imply $x < y$ for any vertices x and y in X.

Proposition 2.11. *Let X be a family. Then :*

(1) *X is series-parallel (N-free), A-free and $\forall$-free.*
(2) *If X is not connectied, then $X = X' + \bar{n}$, where X' is a connected family and $\bar{n}$ is an antichain with n elements.*
(3) *Every subposet of X is a family.*

Proof: The proof of (1) is taken from [6] and that of (3) is from [8]. Also, (2) is clear.

3. A Stanley-Daykin-Paterson series

In this section we will study SDP-series and GSDP-series, and we will show our main theorems. Throughout this section we assume X and Y are finite posets with $|X| = |Y|$.

Definition 3.1. Let x be a vertex of a poset X, and let i be an element of a chain $A_n = \{1, 2, \ldots, n\}$. Then the polynomial defined by

$$F(K_j; X; x; i; n) = \sum_{n=i}^{\infty} N(K_j; X; x; i; n) z^n$$

is said to be a *Stanley-Daykin-Paterson series* (SDP-series, for short) if

$$N(K_j\ ;\ X\ ;\ x\ ;\ i\ ;\ n)$$

is the number of order preserving maps $\omega : X \to A_n$ in K_j such that $\omega(x) = i$.

Definition 3.2. Let x and y be vertices of X and Y, respectively. Then we will define the followings :

(1) (X, x) and (Y, y) are said to be SDP-equivalent in K_j if $F(K_j; X; x; i; n) = F(K_j; Y; y; i; n)$ for all i.
(2) (X, x) and (Y, y) are said to be SDP-congruent in K_j if $F(K_j; X; x; i; n) = F(K_j; Y; y; i; n)$ for some i.
(3) (X, x) and (Y, y) are said to be SDP-similar in K_j if there exists p and q such that

$$\text{the coefficients of } z^k \text{ in } F(K_j; X; x; p; n)$$
$$= \text{the coefficients of } z^k \text{ in } F(K_j; Y; y; q; n)$$

for all $k \geq \max\{p, q\}$.

From Definition 3.2 we easily get the followings :

Proposition 3.3. *Let x and y be vertices of X and Y, respectively. Then :*

(1) *If (X, x) and (Y, y) are SDP-equivalent in K_j, then they are SDP-congruent in K_j (for any j).*

(2) *If (X, x) and (Y, y) are SDP-congruent in K_j, then they are SDP-similar in K_j (for any j).*

Proposition 3.4. *Let x and y be vertices of X and Y, respectively. Then (X, x) and (Y, y) are SDP-congruent in K_j for all $j = 3,\ 4,\ 6$.*

Proof: Note that $F(K_j; X; x; i; n) = 0 = F(K_j; Y; y; i; n)$ for some $i > \max\{|X|,\ |Y|\}$.

For example let X and Y be posets with two elements having the following Hasse diagrams :

X Y

(1) For any $j = 3,\ 4$ and 6, (X, x) and (Y, y) are SDP-congruent in K_j by Proposition 3.4, but (X, x) and (Y, y) are not SDP-equivalent in K_j.

(2) For any $j = 1$ and 5, (X, x) and (Y, y) are SDP-congruent in K_j since $F(K_j; X; x; 1; n) = F(K_j; Y; y; 1; n)$, but (X, x) and (Y, y) are not SDP-equivalent in K_j.

(3) (X, x) and (Y, y) are neither SDP-congruent nor SDP-similar in K_2.

Hence (X, x) and (Y, y) are neither SDP-congruent nor SDP-similar in K_j for any j.

Proposition 3.5. *Suppose that (X, x) and (Y, y) are SDP-equivalent in K_j for some j, where $j = 3,\ 4,\ 5$. Then they are SDP-equivalent in K_6.*

Proof: Note that

$$F(K_6; X; x; i; n) = 0 = F(K_6; Y; y; i; n)$$

or

$$\begin{aligned} F(K_6; X; x; i; n) &= \text{ the coefficient of } z^m \text{ in } F(K_j; X; x; i; n) \\ &= \text{ the coefficient of } z^m \text{ in } F(K_j; Y; y; i; n) \\ &= F(K_6; Y; y; i; n) \end{aligned}$$

where $m = |X|$ and $j = 3,\ 4,\ 5$. Hence they are SDP-equivalent in K_6.

Let X and Y be two posets which are isomorphic. Then it is clear that there exist x in X and y in Y such that (X, x) and (Y, y) are SDP-equivalent in K_j for all $j = 1,\ 2,\ \ldots,\ 6$. But Example 2.4 shows that the converse is not true. Now we will study posets such that the converse is true.

Proposition 3.6. *Suppose that (X, x) and (Y, y) are SDP-equivalent in K_j for some j, where $j = 3,\ 4,\ 5,\ 6$. Then $f_1(x) = f_1(y)$, $f_2(x) = f_2(y)$ and $f_3(x) = f_3(y)$, where $f_3(x)$ is the number of elements which are incomparable to x.*

Proof: Suppose that (X, x) and (Y, y) are SDP-equivalent in K_6. If $p = f_2(x) < f_2(y) = q$, then $F(K_6; X; x; p+1; n) \neq 0 = F(K_6; Y; y; p+1; n)$, which is a contrdiction. Hence $f_2(x) = f_2(y)$. If $p = f_1(x) < f_1(y) = q$, then $F(K_6; X; x; m-p; n) \neq 0 = F(K_j; Y; y; m-p; n)$, where $m = |X|$. This is a contridiction. Hence $f_1(x) = f_1(y)$. Thus $f_3(x) = |X| - f_1(x) - f_2(x) = |Y| - f_1(y) - f_2(y) = f_3(y)$. Therefore it follows from Proposition 3.5 that Proposition 3.6 holds.

Proposition 3.7. *If (X, x) and (Y, y) are SDP-equivalent in K_2 or K_4, then $h(X) = h(Y)$, where $h(X)$ is the height of X.*

Proof: Suppose that $p - 1 = h(X) < h(Y) = q - 1$. Then there exists an i such that the coefficient of z^p in $F(K_j; X; x; i; n) \neq 0 =$ the coefficient of z^p in $F(K_j; Y; y; i; n)$ for some i $(j = 2,\ 4)$, which is a contradiction. Hence $h(X) \geq h(Y)$. Similarly $h(Y) \geq h(X)$. Therefore $h(X) = h(Y)$.

Corollary 3.8. *Suppose that X is a chain or an antichain. If (X, x) and (Y, y) are SDP-equivalent in K_j for some j $(j = 2,\ 4)$, then Y is isomorphic to X.*

Proof: By Proposition 3.7 we have $h(X) = h(Y)$. If X is a chain, then $h(Y) = h(X) = |X| - 1 = |Y| - 1$. Hence Y is a chain. Also, if X is an antichain, then $h(Y) = h(X) = 0$. Hence Y is an antichain.

Theorem 3.9. *Suppose that X is a P–graph of height ≤ 1. If (X, x) and (Y, y) are SDP-equivalent in K_j for each j, then X and Y are isomorphic.*

Proof: If X is a chain or an antichain, then it is clear by Corollary 3.8. Otherwise, let $X = \bar{n_1} \oplus \bar{n_2}$ be a P–graph, where $\bar{n}$ is an antichain with n elements. Assume that x is a maximal element in X. Then by Proposition 3.6 $f_1(x) = f_1(y) = 0$, $f_2(x) = f_2(y) = n_1$ and $f_3(x) = f_3(y) = n_2 - 1$. Hence y is a maximal element in Y. Let Y_1 be the set of all elements which are less than y. Then by Proposition 3.7 $h(Y) = h(X) = 1$ and hence Y_1 is a set of minimal elements. Hence Y_1 is an antichain and $|Y_1| = n_1$. Also, Y is connected since $F(K_6;X;x;t;n) = F(K_6;Y;y;t;n)$, where $t = n_1 + n_2$. If there is a minimal element u of Y such that u is not in Y_1, then it contradicts the fact that $F(K_6;X;x;s;n) = F(K_6;Y;y;s;n)$, where $s = n_1 + 1$. So Y_1 is the set of all minimal elements of Y. Hence the set $Y_2 = Y \backslash Y_1$ is the set of all maximal elements since $h(Y) = 1$. If $Y \neq Y_1 \oplus Y_2$, then it contradidicts the fact that the coefficient of z^t in $F(K_6;X;x;t;n) =$ the coefficient of z^t in $F(K_6;Y;y;t;n)$, where $t = |X|$. Therefore $Y = Y_1 \oplus Y_2 \cong X$. Similarly, we can prove $X \cong Y$ if x is a minimal element. Therefore, X and Y are isomorphic.

Now we will generalize the concept of SDP-series.

Definition 3.10. Let x_k be a vertex of a poset X, and let i_k be an element of a chain $A_n = \{1, 2, \ldots, n\}$. Then the polynomial defined by

$$F(K_j;X;x_k;i_k;n) = \sum_{n=i}^{\infty} N(K_j;X;x_k;i_k;n)z^n, \quad i \text{ is the maximum of } i_k's$$

is said to be a generalized Stanley-Daykin-Paterson series (GSDP-series, for short) if $N(K_j;X;x_k;i_k;n)$ is the number of order preserving maps $\omega : X \to A_n$ in K_j such that $\omega(x_k) = i_k$ for each k.

Definition 3.11. Let x_k and y_k be vertices of posets X and Y, respectively. Then two posets (X, x_k) and (Y, y_k) are said to be GSDP-equivalent in K_j if

$$F(K_j;X;x_k;i_k;n) = F(K_j;Y;y_k;i_k;n)$$

for all i_k.

From Definition 3.11 we know that if (X, x_k) and (Y, y_k) are GSDP-equivalent in K_j, then they are SDP-equivalent in K_j for each k.

Theorem 3.12. *Let $\{x_, \ldots, x_n\}$ be a maximal chain of a P–graph $X = \bar{m}_1 \oplus \cdots \oplus \bar{m}_n$ and let $\{y_1, \ldots, y_n\}$ be a maximal chain of Y with $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$. Suppose that $(X; x_1, \ldots, x_n)$ and $(Y; y_1, \ldots, y_n)$ are GSDP-equivalent in K_j for all $j = 1, \ldots, 6$. Then X and Y are isomorphic.*

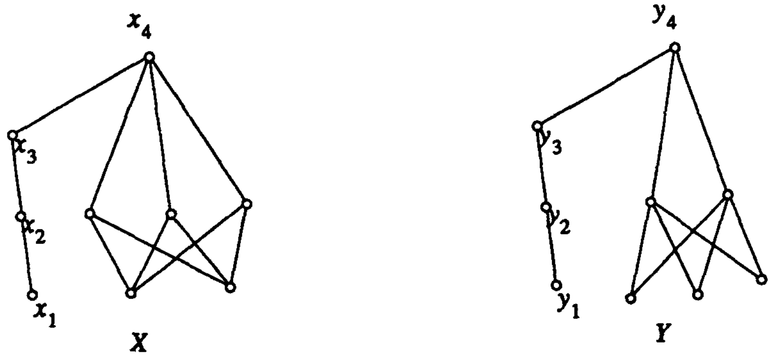
Proof: If $h(X) \leq 1$, then it is obvious by Theorem 3.9. Assume that is holds for $h(Y) < n - 1$. Now we will show that X and Y are isomorphic for $h(X) = n - 1$. Note that $f_1(x_k) = f_1(y_k)$, $f_2(x_k) = f_2(y_k)$ and $f_3(x_k) = f_3(y_k)$ for each k.

Let $I = \{a_1, \ldots, a_p\}$ and $J = \{b_1, \ldots, b_q\}$ be the set of all incomparable elements to y_n and y_{n-1}, respectively. Then since $h(X) = h(Y)$ I and J are antichains with $p = m_n - 1$ and $q = m_{n-1} - 1$. Also $a_i > b_j$ for any i and j. Otherwise, it contradicts the fact that the coefficient of z^s in $F(K_4; X; x_n; t; x_{n-1}; u; m)$ = the coefficient of z^s in $F(K_4; Y; y_n; t; y_{n-1}; u; m)$, where $s = n + m_n + m_{n-1} - 2$, $t = s$ and $u = n - 1$. Hence if $Y_n = I \cup \{y_n\}$, then $Y = (Y - Y_n) \oplus Y_n$. Note that $(X'; x_1, \ldots, x_{n-1})$ and $(Y - Y_n; y_1, \ldots, y_{n-1})$ are GSDP-equivalent in K_j for all $j = 1, \ldots, 6$, where X' is a P–graph $\bar{m}_1 \oplus \cdots \oplus r\bar{m}_{n-1}$. By our induction on $h(x)$ we have $Y - Y_n \cong \bar{m}_1 \oplus \cdots \oplus \bar{m}_{n-1}$. Therefore X and Y are isomorphic.

Theorem 3.13. *Let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ be maximal chains of families X and Y with $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$ respectively, where $n - 1 = h(X)$. Suppose that $(X; x_1, \ldots, x_n)$ and $(Y; y_1, \ldots, y_n)$ are GSDP-equivalent in K_j for all $j = 1, \ldots, 6$. Then X and Y are isomorphic.*

Proof: If $h(X) = 0$, then it follows from Corollary 3.8 that $X \cong Y$. Let $h(X) = 1$. If X is connected, then by Proposition 2.11 and Proposition 2.9 X is a P–graph. Hence by Theorem 3.9 $X \cong Y$. If X is not connected, then by Proposition 2.11 and Proposition 2.9 $X = \bar{n}_1 \oplus \bar{n}_2 + \bar{n}_3$, and $Y = \bar{m}_1 \oplus \bar{m}_2 + \bar{m}_3$ where $\bar{n}$ is an antichain with n elements. By Proposition 3.6, we get $m_1 = n_1$, $m_2 = n_2$ and $m_3 = n_3$. Thus $X \cong Y$ if $h(X) = 1$.
Now, assume that it holds for $h(X) < n - 1$, where $n \geq 3$. We will show this by induction. Assume that $h(X) = n - 1$. If X is connected, we have $X = A \oplus B$ for some family subposets A and B. Let P be the set of all minimal elements in B, and let Q be the set of all maximal elements in A. Then there is $p \in \mathbf{N}$ such that $x_p = Q$ and $x_{p+1} \in P$. Hence by Proposition 3.6 we have $f_1(y_p) = f_1(x_p) = |B|$, $f_2(y_p) = f_2(x_p)$ and $f_3(y_p) = f_3(x_p) = |Q| - 1$, $f_1(y_{p+1}) = f_1(x_{p+1})$, $f_2(y_{p+1}) = f_2(x_{p+1}) = |A|$ and $f_3(y_{p+1}) = f_3(x_{p+1}) = |P| - 1$. If $M = \{y\ :\ y \text{ is an upper cover of } y_p\}$ and $N = \{y\ :\ y \text{ is a lower cover of } y_{p+1}\}$, then by the definition of M and N they are antichains. Let $A' = \{x \in Y\ :\ x < y_{p+1}\}$ and $B' = \{x \in Y\ :\ x > y_p\}$. Then $A' \cap B' = \emptyset$ since y_{p+1} is an upper cover of y_p. Hence $|A' \cup B'| = |A'| + |B'| = f_2(y_{p+1}) + f_1(y_p) = f_2(x_{p+1}) + f_1(x_p) = |A| + |B|$ and so $|A' \cup B'| = |X| = |Y|$. So $A' \cup B' = Y$. Let P' be the set of all minimal elements in B' and Q' be the set of all maximal elements in A'. Since X is N- free, we have $Q' \oplus P'$. Thus $A' \oplus B' = Y$. By our induction $A' \cong A$ and $B' \cong B$. Therefore $X \cong Y$. If X is not connected, then by Proposition 2.11 $X = X' + \bar{m}$ and $Y = Y' + \bar{n}$ for some m and n. Hence $m = n$ and $(X'\ ;\ x_1,\ \ldots,\ x_n)$ and $(Y'\ ;\ y_1,\ \ldots,\ y_n)$ are GSDP-equivalent in K_j for all $j = 1, \ldots, 6$. It can be proved by the same way as in the connected case. Thus $X \cong Y$. Therefore, Theorem 3.13 holds.

If we replace families by (connected) series-parallel posets in Theorem 3.13, then it is not true that X and Y are isomorphic. For example, let X and Y be posets with following Hasse diagrams :



Then $(X\ ;\ x_1,\ \ldots,\ x_4)$ and $(Y\ ;\ y_1,\ \ldots,\ y_4)$ are GSDP-equivalent in K_j for all $j = 1,\ \ldots,\ 6$, but X and Y are not isomorphic.

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