한국수학교육학회지〈수학교육〉
1992. 6. 제 31 권 제 1 호 49-51

Journal of the Korea Society of Mathematical Education
June. 1992; Vol. 31, No. 1, 49-51

On γ^*- Spaces Over α

Won Keun Min

Department of Mathematics, Kangweon
National University ,Chuncheon, 200-701, Korea

I. Introduction and Preliminaries

In this paper, we introduce the concept of γ^*–space over α and show that some results which hold for γ^*–space can be extended to γ^*–space over α.

Let (X, τ) be a topological space and let g be a function from $\alpha \times X$ (where α is the infinite initial ordinal number) to τ such that

$$x \in g(\beta, x) \qquad \text{for all} \qquad \beta < \alpha,$$

$$g(\beta, x) \subset g(\gamma, x) \quad \text{for} \quad \gamma < \beta < \alpha \quad \text{and for every} \quad x \in X.$$

We call such a map an αoc–map (α–open covering map). In the particular case of $\alpha = \omega$, the function g is called *coc*–map. Note that if we let $G_\beta = \{g(\beta, x) : x \in X\}$, then $\{G_\beta : \beta < \alpha\}$ is a net of open covers of X such that G_β refines G_γ for $\gamma < \beta$.

For any subset S of X, we define

$$g(\beta, S) = \cup\{g(\beta, x) : x \in S\},$$

$$g^2(\beta, S) = g(\beta, g(\beta, S)).$$

Let A, B be families of subsets of X. Consider the following separation properties on an αoc–map g. For each $U \in A$, $V \in B$, if there exists a $\beta < \alpha$ such that

(1) $U \cap g(\beta, V) = \emptyset$, then g separates B from A.
(2) $U \cap g^2(\beta, V) = \emptyset$, then g separates doubly B from A.
(3) $U \cap g(\beta, V)^- = \emptyset$, then g separates regularly B from A.
(4) $g(\beta, U) \cap g(\beta, V) = \emptyset$, then g separates disjointly B from A.
(5) $U \cap S_t(V, G_\beta) = \emptyset$, then g separates starly B from A.

II. Definition and Theorem

It is well known that a space is γ^*–space if it has a *coc*–map satisfying : if $x \in g(n,\ y_n)$ and $y_n \in g(n,\ x_n)$ for each n, then x is a cluster point of $\{x_n\}$. Similarly, we introduce an extension space of γ^*–space.

Definition 2.1. A space X is γ^*–space over α if it has an αoc–map satisfying : if $x \in g(\beta,\ y_\beta)$ and $y_\beta \in g(\beta,\ x_\beta)$ for each $\beta < \alpha$, then x is a cluster point of $\{x_\beta\}$. And we call that g is γ^*–function over α. A space which is a γ^*–space over ω is said to be γ^*–space.

Definition 2.2. A space is developable over α if and only if it has an αoc–map satisfying : if $x,\ x_\beta \in g(\beta,\ y_\beta)$ for $\beta < \alpha$, then x is a cluster point of $\{x_\beta\}$. A space which is developable over ω is said to be developable.

Theorem 2.3. *A space γ^*–space over α if and only if it has an αoc–map which separates doubly closed sets from points.*

Proof: Let g be an αoc–map satisfying the condition in (2.1). Let F be a closed set not containing x. Suppose that $x \in g^2(\beta,\ F)$ for every $\beta < \alpha$. Then there are $x_\beta \in F$ and $y_\beta \in X$ so that x is a cluster point of $\{x_\beta\}$. This contradicts $x \notin F$.

Conversely, let g separates doubly closed sets from points and let $x \in g(\beta,\ y_\beta)$ and $y_\beta \in g(\beta,\ x_\beta)$. If x is not a cluster point of $\{x_\beta\}$, $F = \{x_\beta\}^-$ is a closed set not containing x. Since g separates doubly closed sets from points, there exists $\gamma < \alpha$ so that $x \notin g^2(\gamma,\ F)$. However, $x \in g(\gamma,\ y_\gamma)$ and $y_\gamma \in g(\gamma,\ x_\gamma)$ with $x_\gamma \in F$, which is a contradiction.

Corollary 2.4. *A space is a γ^*–space if and only if it has a coc–map which separates doubly closed sets from points.*

Theorem 2.5. *A space is a developable space over α if and only if it has an $\alpha \circ c$–map which separates starly points from closed sets.*

Proof: Let g be an αoc–map and F be a closed set not containing x. Then $S_t(x,\ G_\beta) \cap F \neq \emptyset$, where $G_\beta = \{g(\beta,\ x)\ :\ x \in X,\ \beta < \alpha\}$, if and only if there are x_β and y_β such that $x_\beta,\ x \in g(\beta,\ y_\beta)$. By the same argument as employed in the Theorem 2.3. (4) by Sakong [5], we are done.

Corollary 2.6. *A space is a developable space if and only if it has a coc–map which separates starly points from closed sets.*

It is well known that a space is K-semistratifiable over α if and only if it has an αoc-map which separates closed sets from compact sets and semistratifiable over α if and only if it has an αoc-map which separates closed sets from points (see [4]).

Theorem 2.7. *For an $\aleph_1$-compact space X. If X is K-semistratifiable over α, then X is a γ^*-space over α.*

Proof: Let g be an αoc-map on X separating closed sets from compact sets. Suppose that there exists $x \notin F$ with F closed and $x \in g^2(\beta, F)$ for all $\beta < \alpha$. Then we have net $\{x_\beta\}$ and $\{y_\beta\}$ such that $x \in g(\beta, y_\beta)$, $y_\beta \in g(\beta, x_\beta)$ and $x_\beta \in F$. Since $\{y_\beta\}$ converges to x, $K = \{x\} \cup \{y_\beta : \beta < \alpha\}$ is compact. We may assume that $K \cap F = \emptyset$. As g separates closed sets from compact sets, there exists $\gamma < \alpha$ such that $g(\gamma, F) \cap K = \emptyset$. However, $y_\gamma \in g(\gamma, x_\gamma) \cap K$, which is a contradiction.

Corollary 2.8. *For a compact space X. If X is a K-semistratifiable space, then X is a γ^*-space.*

We recall that a space is γ-space over α if and only if it has an αoc-map g astisfying : if $x_\beta \in g(\beta, y_\beta)$ and $y_\beta \in g(\beta, x)$ for each $\beta < \alpha$, then x is a cluster point of $\{x_\beta\}$ (see [4]).

Theorem 2.9. *If a space is γ^*-space over α and a γ-space over α, then it is a semistratifiable γ-space over α. Moreover, it is a developable space over α.*

Proof: Let g be an αoc-map such that $x,\ x_\beta \in g(\beta, x_\beta)$. Since g is the function of semistratifiable over α, $\{y_\beta\}$ converges to x. Now, we may assume that $y_\beta \in g(\beta, x)$. Since g is γ-function over α, $\{x_\beta\}$ clusters to x. Therefore, g is a developable function over α.

Corollary 2.10. *If a space γ^*-space and a γ-space, then it is a semistratifiable space. A semistratifiable γ-space is developable.*

References

1. J. G. Ceder, *Some generalizations of metric spaces*, Pacific J. Math. **11** (1961), 105–125.
2. K. L. Lee, *Linearly semistratifiable spaces*, J. Korea Math. **11.** no. 1 (1974).
3. D. J. Lutzer, *Semimetrizable and Stratifiable spaces*, Gen. Top. Appi **1** (1971), 43–48.
4. W. K. Min, *Charactization of linearly K-semistratifiable*, The Journal of Science and Technology **27** (1988).
5. J. S. Sakong, *Generalized metric spaces defined by seperation properties*, Ph. D. Thesis, 1983.
6. J. E. Vaughan, *Linearly stratifiable spaces*, Pacific J. Math. **43** (1972), 253–266.