한국수학교육학회지(수학교육)
1992. 6. 제 31 권 제 1 호 53-58

Journal of the Korea Society of Mathematical Education
June. 1992: Vol. 31, No. 1, 53-58

Approximate Max-Min Controllability for Delay System

Kwun, Young Chel

Department of Mathematics, Dong-A University, Pusan, 604-714 Korea

1. Introduction

Let X and U be a reflexive Banach spaces over C of R, with norms $\|\cdot\|$ and $\|\cdot\|'_U$ respectively.

We consider an abstract control system (1) on X with time-delays;

$$\begin{cases} \frac{dx(t)}{dt} = A_0x(t) + \int_{-h}^{0} d\eta(s)x(t+s) + B(t)u(t) \quad \text{a.e. } t > 0 \\ x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e. } s \in [-h,\ 0), \end{cases} \tag{1}$$

where $g = (g^0,\ g^1) \in X \times L_q([-h,\ 0]; X)$, $u \in L_p^{\text{loc}}(R^+; U)$, $p,\ q \in (1, \infty)$, $\{B(t);\ t \geq 0\} \subset \mathcal{L}(U, X)$ is a bounded operators from U into X, A_0 generates a C_0-semigroup $\{T(t);\ t \geq 0\}$ on X and η is a Stieltjes measure. Let $W(t)$ be the fundamental solution of (1), which is a unique of the equation

$$W(t) = \begin{cases} T(t) + \int_0^t T(t-s) \int_{-h}^{0} d\eta(\xi)W(\xi+s)ds & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then $W(t) \in \mathcal{L}(X)$ for each $t \geq 0$ and $W(t)$ is strongly continuous in R^+ (e.g. Nakagiri [2]).

If the condition

$$A_I(\cdot) \in L_{q'}([-h,0];\ \mathcal{L}(X)), \quad 1/q + 1/q' = 1 \tag{2}$$

is satisfied, then for each $t \geq 0$, the operator valued function $U_t(\cdot)$ given by

$$U_t(s) = \int_{-h}^{s} W(t-s+\xi)d\eta(\xi) \quad \text{a.e. } s \in [-h,0] \tag{3}$$

belongs to $L_{q'}([-h,0];\ \mathcal{L}(X))$. This follows from the Hausdorff-Young inequality. Hence the function

$$x(t;\ g,\ u) = \begin{cases} W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds + \int_0^t W(t-s)B(s)u(s)ds, & t \geq 0 \\ g^1(t) \quad \text{a.e. } t \in [-h,0) \end{cases} \tag{4}$$

is well-defined and is an element of $C(R^+;X)$. Moreover it is proved in [2] that under the condition (2), the function $x(t) = x(t;\ g,\ u)$ is a unique solution of the integrated form of (1) by $T(t)$, i.e.,

$$\begin{aligned} x(t) =& T(t)g^0 + \int_0^t T(t-s)B(s)u(s)ds \\ &+ \int_0^t T(t-s)\int_{-h}^{0} d\eta(\xi)x(s+\xi)ds, \quad t \geq 0. \end{aligned} \tag{5}$$

In this sense, this function $x(t)$ is called the mild solution of (1). In the system (1), $u(t)$ and $g^1(s)$ are called a forcing function control and initial function control, respectively.

In [1], W. L. Chan and C. W. Li consider the max-min controllability in linear pursuit games with the norm bounded controls for linear evolution equation.

In [3], J. Y. Park, S. Nakagiri and M. Yamamoto study the admissible controllability for the system (1).

In [4], Y. C. Kwun consider the max-min controllability for the system (1).

The purpose of this paper is to prove the approximate max-min controllability results for the abstract control system (1), we follow the method used in [4].

2. Approximate Max-Min Controllability

For each $t > 0$, $\delta > 0$, $\rho > 0$ and $p,\ q \in (1,\infty)$, we define the constraint sets

$$U_p^\delta = \left\{ u \in L_p([0,t_1];\ U);\ \|u\|_p = \left(\int_0^{t_1} \|u(s)\|_U^p ds\right)^{1/p} \leq \delta \right\}$$

and

$$G_q^\rho = \left\{ g^1 \in L_q([-h,0];\ X);\ \|g^1\|_q = \left(\int_{-h}^{0} \|g^1(s)\|^q ds\right)^{1/q} \leq \rho \right\}$$

For convenience, we denote the above linear differential game problem by the notation

$$(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T = [0,\ t_1]).$$

For each $t > 0$ we define two operators $\mathcal{B}_{t_1}$; $L_p([0,t_1];\ U) \to X$ and $\mathcal{C}_{t_1}$; $L_q([-h,0];\ X) \to X$ by

$$\mathcal{B}_{t_1}u = \int_o^{t_1} W(t_1 - s)B(s)u(s)ds$$

and

$$\mathcal{C}_{t_1}g^1 = \int_{-h}^{0} U_{t_1}(s)g^1(s)ds$$

respectively.

Lemma 2.1 ([3]). *We assume that $T(t)$ is compact for all $t > 0$. Then the operators $\mathcal{B}_t$ and $\mathcal{C}_t$ are continuous linear compact operators.*

Lemma 2.2 ([3]). *The sets $\mathcal{B}_t(U_p^\delta)$, $\mathcal{C}_t(G_q^\rho)$, R are compact and convex.*

Definition 2.1. The system $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ is said to be approximate max-min controllable if for each initial function control $g^1 \in G_q^\rho(T)$, there exists a forcing function control $u \in U_p^\delta(T)$ such that $x(t_1;\ (g^0, g^1),\ u) \in B(0;\varepsilon)$ where $B(0;\varepsilon) = \{x \in X;\ \|x\| < \varepsilon\}$.

Lemma 2.3 ([3]). *Let E and F be closed convex sets in X. Then $E \subset F$ if and only if*

$$\sup_{x\in E}\langle x,\ x^*\rangle \le \sup_{x\in F}\langle x,\ x^*\rangle$$

for all $x^* \in X^*$.

Now, we are going to consider problems analogous to that in [4]. Since the results are similar, we shall therefore only sketch the proofs for brevity.

Theorem 2.1. *The system $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ is approximate max-min controllable if and only if for every $x^* \in X^*$, we have*

$$|\langle W(t_1)g^0,\ x^*\rangle| \le \delta\left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\right]^{1/p'} - \rho\left[\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'} ds\right]^{1/q'} + \varepsilon\|x^*\|_* \qquad (6)$$

Proof: By Definition 2.1, the system $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ is approximate max-min controllable if and only if

$$W(t_1)g^0 + \mathcal{C}_{t_1}(G_q^\rho(T)) \subset -\mathcal{B}_{t_1}(U_p^\delta(T)) + B(0;\varepsilon). \qquad (7)$$

Since $\mathcal{C}_{t_1}(G_q^\rho(T))$ is closed, the set $W(t_1)g^0 + \mathcal{C}_{t_1}(G_q^\rho(T))$ is closed in X. Moreover

$$-\mathcal{B}_{t_1}(U_p^\delta(T)) + B(0,\ \varepsilon) = \mathcal{B}_{t_1}(U_p^\delta(T)) + B(0,\ \varepsilon)$$

is closed. In fact, let $u_n = y_n + z_n$, $y_n \in \mathcal{B}_{t_1}(U_p^\delta(T))$ and $z_n \in B(0,\ \varepsilon)$ $(n \geq 1)$ and let u_n converge to u strongly in X. Then by the reflexiveness of X, there exists a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z$ weakly for some $z \in X$ as $k \to \infty$. Thus $y_{n_k} = u_{n_k} - z_{n_k}$ converges weakly to $y = u - z$ as $k \to \infty$. Since the convex set $\mathcal{B}_{t_1}(U_p^\delta(T))$ is closed, we have $y \in \mathcal{B}_{t_1}(U_p^\delta(T))$. On the other hand, since $z_{n_k} \to z$ weakly, we have

$$\|z\| \leq \lim_{k\to\infty} \inf \|z_{n_k}\| \varepsilon,$$

namely, $z \in B(0;\ \varepsilon)$. Therefore $u \in \mathcal{B}_{t_1}(U_p^\rho(T)) + B(0;\ \varepsilon)$. Obviously both $W(t_1)g^0 + \mathcal{C}_{t_1}(G_q^\rho(T))$ and $\mathcal{B}_{t_1}(U_P^\delta(T)) + B(0;\ \varepsilon)$ are convex. Thus we can apply Lemma 2.3, so that we see that (7) is equivalent to that

$$\begin{aligned} &\sup\{\langle W(t_1)g^0 + \mathcal{C}_{t_1}g^1,\ x^*\rangle;\ g^1 \in G_q^\rho(T)\} \\ &\quad \leq \sup\{\langle \mathcal{B}_{t_1}u + y,\ x^8\rangle;\ u \in U_p^\delta(T),\ \ y \in B(0;\ \varepsilon)\} \end{aligned} \tag{8}$$

for each $x^* \in X^*$. Then, similar method as in Theorem 2.1 of [4],

$$\begin{aligned} \langle W(t_1)g^0,\ x^*\rangle \leq \delta & \left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\right]^{1/p'} \\ & - \rho\left[\int_{-h}^0 \|U_{t_1}^*(s)x^*\|_*^{q'} ds\right]^{1/q'} + \varepsilon\|x^8\|_*. \end{aligned} \tag{9}$$

Replacing x^* by $-x^*$ in (9), we obtain the condition (6). This completes the proof.

Theorem 2.2. *If the system $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ is approximate max-min controllable, then there exists a minimal time interval $T_{\hat{t}} = [0,\ t_1]$, over which the system $(g^0,\ U_p^\delta(T_t),\ G_q^\rho(T_{\hat{t}}),\ T_{\hat{t}})$ remains to be approximate max-min controllable and there exists $x^* \in X^*$, with $\|\hat{x}^*\|_* = 1$, such that the following equality holds;*

$$\begin{aligned} |\langle W(\hat{t})g^0,\ \hat{x}^*\rangle| = \delta & \left[\int_0^{\hat{t}} \|B^*(s)W^*(\hat{t} - s)\hat{x}^*\|_{U^*}^{p'} ds\right]^{1/p'} \\ & - \rho\left[\int_{-h}^0 \|U_{\hat{t}}^*(s)\hat{x}^*\|_*^{q'} ds\right]^{1/q'} + \varepsilon\|\hat{x}^*\|_* \end{aligned} \tag{10}$$

On the otherhand, if the equality in (10) cannot hold for every $x^* \in X^*$, then $T = [0,\ t_1]$ is not a minimal time interval for system.

Proof: It is omitted.

Theorem 2.3. *Suppose that the system* $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ *is approximate max-min controllable; then, there exists an optimal value* $\hat{\delta}_\varepsilon \geq 0$ *such that the system* $(g^0,\ U_p^{\hat{\delta}_\varepsilon}(T),\ G_q^\rho(T),\ T)$ *remains to be approximate max-min controllable and*

$$\hat{\delta}_\varepsilon = \max\{0,\ \delta_1\} \tag{11}$$

where

$$\delta_1 = \sup\{|\langle W(t_1)g^0,\ x^*\rangle| + \rho[\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'} ds]^{1/q'} - \varepsilon\|x^*\|_* \ ;\ [\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds]^{1/p'} = 1\}. \tag{12}$$

Moreover, under the assumption in Theorem 2.6 of [4], *the supremum in (11) is attained for some* $\hat{x}^* \in X^*$.

Proof: The proofs of the existence of $\hat{\delta}_\varepsilon$ and of the supremum of (11) being attained are similar to those of Theorem 2.2 and Theorem 2.6 of [4]; we only need to prove (12), which quite different from the previous case.

Obviously, $\delta_1 \leq \hat{\delta}_\varepsilon$. We claim that, if $\hat{\delta}_\varepsilon > 0$, then $\delta_1 > 0$ and thus, by a similar argument in Theorem 2.5 of [4], we have $\delta_1 = \hat{\delta}_\varepsilon$, and so (11) holds.

Suppose that $\hat{\delta}_\varepsilon > 0$, but $\delta_1 \leq 0$; then, from (12), we know that, for every $x^* \in X^*$,

$$|\langle W(t_1)g^0,\ x^*\rangle| + \rho\left[\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{1/q'} ds\right]^{1/q'} \leq \varepsilon\|x^*\|_*,$$

which shows that the system $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ is approximate max-min controllable, and so $\hat{\delta}_\varepsilon > 0$ is impossible.

Theorem 2.4. *If the system* $(g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T)$ *is approximate max-min controllable, then there exists a minimal value* $\hat{\varepsilon} \geq 0$, *such that the system is* $\hat{\varepsilon}$*-approximate max-min controllable (i.e.* $x(t_1,\ (g^0, g^1),\ u) \in B(0;\ \hat{\varepsilon})$*) and*

$$\hat{\varepsilon} = \max(0,\ \varepsilon_1)$$

where

$$\varepsilon_1 = \max\{|\langle W(t_1)g^0,\ x^*\rangle| + \rho\left[\int_{-h}^{0} \|U_{t_1}^*(s)x^*\|_*^{q'} ds\right]^{1/q'} - \delta\left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'}\right]^{1/p'} \ ;\ \|x^*\|_* = 1\}. \tag{13}$$

Proof: Consider the set

$$E = \{\varepsilon' \geq 0;\ \text{the system } (g^0,\ U_p^\delta(T),\ G_q^\rho(T),\ T) \text{ is } \varepsilon' - \text{ approximate max-min controllable (i.e. } x(t_1,\ (g^0, g^1),\ u) \in B(0;\varepsilon'))\}$$

Since $E \neq$ set, we can define $\hat{\varepsilon} = \inf E \geq 0$. Now, choose a sequence $\{\varepsilon\} \subset E$ such that $\varepsilon_n \to \hat{\varepsilon}$; and so, for every $x^* \in X^*$, we have

$$|\langle W(t_1)g^0,\ x^*\rangle| \leq \delta \left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\right]^{1/p'} - \rho \left[\int_{-h}^{0} \|U_{t_1}^* x^*\|_*^{q'} ds\right]^{1/q'} + \varepsilon_n \|x^*\|_*.$$

Passing to the limit as $n \to \infty$, we have

$$|\langle W(t_1)g^0,\ x^*\rangle| \leq \delta \left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x^*\|_{U^*}^{p'} ds\right]^{1/p'} - \rho \left[\int_{-h}^{0} \|U_{t_1}^*(s) x^*\|_*^{q'} ds\right]^{1/q'} \hat{\varepsilon} \|x^*\|_*. \tag{14}$$

Thus, $\hat{\varepsilon} \in E$. We see that the right-hand side of (13) is a continuous function of $x^* \in X^*$, so that its maximum exists. Comparing (13) with (14), we see that $\varepsilon_1 \leq \hat{\varepsilon}$. Suppose that $\hat{\varepsilon} > 0$; we can choose a sequence $\{\varepsilon_n'\}$ such that $0 \leq \varepsilon_n' < \hat{\varepsilon}$ and $\varepsilon_n' \to \hat{\varepsilon}$. Thus, for each n, there exists $x_n^* \in X^*$, with $\|x_n^*\|_* = 1$, and

$$|\langle W(t_1)g^0,\ x_n^*\rangle| > \delta \left[\int_0^{t_1} \|B^*(s)W^*(t_1 - s)x_n^*\|_{U^*}^{p'} ds\right]^{1/p'} - \rho \left[\int_{-h}^{0} \|U_{t_1}^*(s) x_n^*\|_*^{q'} ds\right]^{1/q'}$$

Without loss of generality, we may assume that $x_n^* \to \hat{x}^*$, with $\|\hat{x}^*\|_* = 1$. Then passing to the limit as $n \to \infty$, we have $\hat{\varepsilon} \leq \varepsilon_1$, and so the result follows. When $\varepsilon = 0$, the above result reduces to the max-min controllable case.

References

1. W. L. Chan and C. W. Li, *Max-Min Controllability in Pursuit Games with Norm-Bounded Controls*, J. Optim. Theory Appl. **37** no. 1 (1982), 89–113.
2. S. Nakagiri, *Optimal Control of Linear Retarded Systems in Banach Spaces*, J. Math. Anal. Appl. **120** (1986), 169–210.
3. J. Y. Park, S. Nakagiri and M. Yamamoto, *Admissible Controllability for Linear Time-delay Systems in Banach Spaces-A Problem in Game Theorey*, Sci. Papers College Arts Sci Univ. Tokyo **39** no. 1 (1989), 45–55.
4. Y. C. Kwun, *Max-Min Controllability for the Time Delay System*, Pusan Kyongnam Math. J. **7** no. 1 (1991), 29–36.