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圓孔을 갖는 有限板의 應力集中에 대한 P-Version 모델

P-Version Model of Stress Concentration Around a Circular Hole in Finite Strips

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요 지

면내거동과 휨거동을 받는 원형구멍을 갖는 유한평판에서 원형구멍 주위의 응력을 모델링하기 위해 p-version 유한요소법이 제시되었으며 또한, 동일한 문제로써 원형구멍으로 부터 발생된 균열해석을 위해 균열확장법이 사용되었다. 적분형 르장드르함수에 기초한 p-version 유한요소법이 원형구멍 주위의 응력경사가 심한 기하형상을 모델링하는데 적합함을 보여준다. 한편, 원형 경계조건을 표현하는데 이산화오차를 피하기 위해 초유한사상기법이 사용되었다. 앞에서 제시된 방법을 통한 수치해석 결과는 Nisida, Howland, Newman 등의 실험및 이론결과와 종래의 유한요소법에 의한 수치해석결과와 비교하여 우수한 값을 보여 주고있다.

Abstract

This paper presents a p-version finite element approach for modeling the stress distribution around a circular hole in a finite strip subjected to membrane and flexural behaviors. Also, same problem with a crack emanating from a perforated tension strip was solved by virtual crack extension method. The p-version of the finite element method based on integrals of Legendre polynomials is shown to perform very well for modeling geometries with very steep stress gradients in the vicinity of a circular cutout. Here, the transfinite mapping technique for circular boundaries was used to avoid the discretization errors. The numerical results from the proposed scheme have a good comparison with those by Nisida, Howland, Newman etc. and the conventional finite element approach.

1. Introduction

In the aerospace and machine structures, fastner(rivet) holes are inherent stress raisers and thus sources of fatigue cracks. As fastner holes are generally used to connect major structural components, they are often subjected to repeated load spectra. Load transfer occurs between the rivet and the bore of fastner hole generating a typical elastic gross section stress concentration factor of three or greater in tension or bending

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stress fields. As a result of these high stress concentrations, the critical assesement of any crack emanating from these circular holes is therefore of prime importance. In this study, the stress concentration factors of fastner holes subjected to membrane or flexural behaviors and the stress intensity factors of crack tip emanating from fastner holes have been widely investigated. Finite element techniques that are used to compute stresses and displacements in the vicinity of fastner holes are divided into two categories. In the first approach, the solution is then approximated over the given domain by interpolation functions which are piecewise polynomials on convex subdomains and which are globally in C^n, $n \geq 0$, where n depends upon the partial differential equation. The degrees of the approximating piecewise polynomials are fixed (usually at some low number such as 1 or 2) and the accuracy of the approximation is increased by allowing h, the maximum diameter of the finite elements, to go to zero. We refer to this approach as the h-version of the finite element method. The h-version has been studied extensively and asymptotic error bounds as $h \rightarrow 0$ are well known for its rate of convergence. However, in the second approach, the accuracy is increased by allowing the degree p of the piecewise polynomials to go to infinity. We call this approach the p-version of the finite element method. In a number of cases the given domain has reentrant corners or sudden changes occur in the boundary conditions. In the neighborhood of such points, the exact solution is of the form.[1]

$$u(r,\ \theta) = r^{\alpha} \cdot F(\theta) + G(r,\ \theta) \qquad \alpha > 0 \tag{1}$$

$u(r, \theta)$ is the displacement vector, r and θ are polar coordinates centered on the point, α is determined from the condition that the solution must satisfy the Navier-Lame' equations and the boundary conditions on the edges that meet at the corner. It is noted that $G(r, \theta)$ represents the smooth part of the solution and the first term the singular part. Thus, singular behavior may occur when some part of boundaries are not smooth. The sources of singularities may be classified under three headings.[2]

(1) Geometric : reentrant corners, cracks, cutouts with sharp corners,discontinuities in curvature and thickness, presence of stiffeners, mixed boundaries, etc.

(2) Loading : concentrated sources over the surface and at the edges, line sources over the surface, and sudden changes in the intensity of external sources, etc.

(3) Material : sudden changes in material properties, as in the case of laminated materials, incompressible materials, etc.

In this paper, we shall investigate the stress distribution due to the geometric sources in the neighborhood of a circular hole (with/without cracks) in a finite strip subjected to membrane forces and uniformly distributed bending moments. As the stress gradients in the vicinity of a singular point are very steep, the p-version of the finite element method on the basis of Integrals of Legendre polynomials is expected to perform very well for modeling geometries with circular cutouts[2]. In the p-version, as we aware of it, the size of the element is usually large and hence the probability of distortions is more. So, the transfinite mapping for circular boundaries is adopted to avoid the discretization errors.

2. Transfinite Mapping Technique

In this study, we are faced with a nonpolygonal domain due to a circular hole. The eraly approaches in finite element modeling required that the boundary, $\partial\Omega$, of Ω be approximated by a polygonal arc. The Babuska paradox[3] describes an error associated with modeling a curved boundary by straight-sided elements. To avoid this error, it is necessary to find the mapping function which will exactly map the standard element to the sides of the real element including the four corner nodes by making use of the exact geometric parameters of the curved boundary. In this paper, an attempt has been made to generate a curved boundary by using a transfinite mapping technique

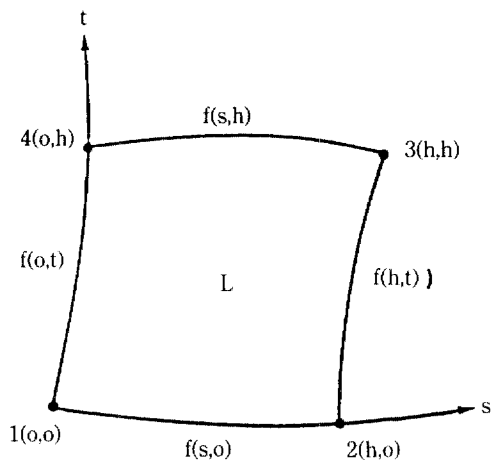

Fig. 1. Domain L in the s-t plane

proposed by Gordon and Hall.[4] Let f be a continuous function of two independent variables with domain L ; [0,h] x [0,h] in the s-t plane as shown in Fig. 1. By a projector P, we mean a linear operator from the linear space T of all continuous bivariate function f, with domain L, onto a subspace of functions. For example, if the operator P_s is defined by the formula:

$$P_s[f]=(1-s/h)\cdot f(0,t)+(s/h)\cdot f(h,t) \quad (2)$$

It can be expresses by the general form as follows:

$$P_s[f]=\sum_{i=0}^{m} f(s_i,\ t)\cdot \Phi_i(s) \quad (3)$$

where $0=s_0<s_1<\cdots<s_m=h$ and

$$\Phi_i(s)=\prod_{j\neq i}(s-s_j)/\prod_{j\neq i}(s_i-s_j),\ 0\leq i\leq m \quad (4)$$

are the fundamental functions for Lagrange polynomial interpolations. For completeness and later reference, we display the analogous formula for P_t:

$$P_t[f]=\sum_{j=0}^{n} f(s,\ t_j)\cdot \psi_j(t) \quad (5)$$

where $0=t_0<t_1<\cdots<t_n=h$ and

$$\psi_j(t)=\prod_{i\neq j}(t-t_i)/\prod_{i\neq j}(t_j-t_i),\ 0\leq j\leq n \quad (6)$$

There is a way to compound the projectors P_s and P_t by using Boolean sum.

$$P_s\oplus P_t\equiv P_s+P_t-P_s\cdot P_t \quad (7)$$

3. P-convergence Approximations in Fracture Mechanics

The rate of h-convergence for problems involving singularities has been first examined by Tong et al.[5] and Johnson et al.[6]. However, a considerable amount of numerical experience has been compiled in the last ten year.[7,8] It has been fouhd that the exact value of the strain energy U is determined by extrapolation. The procedure is based on the theorem which establishes the rate of convergence for the p-version and the corresponding inverse theorem as follows[8] :

$$U(u-u_p)=U(u)-U(u_p)\leq \frac{k}{N_p^{2\alpha}} \quad (8)$$

in which U is the strain energy; u is the exact solution; u_p is the finite element solution by the p-version, using uniform p-distribution; k is a constant, which depends on the domain, the loading, the mesh, and ; Np is the number of degrees after the boundary conditions were enforced; α is the smoothness index of the exact solution u which can be determined by the solid angles at the corners of the domain, the boundary conditions and Poisson's ratio. When N is sufficiently large so that the less or equal sign in Eq.(8) can be replaced by an equals sign, we say that the approximation is in the asymptotic range. U(u) can be determined by extrapolation[3] :

$$U(u)\simeq \frac{U(u_p)\cdot N_p^{2\alpha}-U(u_{p-1})\cdot N_{p-1}^{2\alpha}}{N_p^{2\alpha}-N_{p-1}^{2\alpha}} \quad (9)$$

In this way, the exact value of the strain energy can be computed with high precision(four or five significant digits) without knowing the exact solution u. Evidently, the extrapolation formula yields good estimates even at low p-values, say p=5.

4. Computation of Stress Intensity Factors

The finite element method has been used by a number of investigators to determine elastic stress intensity factors for cracked bodies. The characteristic elastic square root singularity has been represented by the use of virtual crack extension method in this work. This method is one of the energy methods in which the stress intensity factors are obtained from the measurement of the energy release rate, that is, the energy difference between succesive crack tip positions along a mesh. In the virtual crack extension method, the energy release rate G is computed from both displacement field before the crack extension and the change of stiffnes during the extension, as described in equation

$$G=-\frac{d\Pi}{da}=-\frac{1}{2}\{u\}^T\frac{\partial[K]}{\partial a}\{u\}+\{u\}^T\frac{\partial[f]}{\partial a} \quad (10)$$

where Π is the total potential energy, a is the crack length, {u} is the displacement vector before crack extension, [K] is the stiffness of the finite element array, and [f] is the load vector. If we consider a virtual increase Δa in crack length with no change in external mechanical or thermal loads, G is given by

$$G=-\frac{1}{2}\{u\}^T\frac{\partial[K]}{\partial a}\{u\} \quad (11)$$

However, since the quantity G is very sensitive to the crack length increment Δa, the sensitivity test was investigated between G and Δa in Fig. 2. From this figure Δa was adopted by $10^{-7}a$ and Poisson's ratio was 0.3. It is noted that the linearity of the strain energy release rate versus the reciprocal of the number of degrees of freedom is very strong as shown in Fig. 3. To illustrate this, we computed estimetes of G by means of linear extrapolation. When three succesive finite solutions are in asmptotic range, one can get the extrapolation formula using Eq.(9) as follows:

$$\frac{\mathrm{LOG}\ \dfrac{G_{ex}-G_p}{G_{ex}-G_{p-1}}}{\mathrm{LOG}\ \dfrac{G_{ex}-G_{p-1}}{G_{ex}-G_{p-2}}}=\frac{\mathrm{LOG}\ \dfrac{N_{p-1}}{N_p}}{\mathrm{LOG}\ \dfrac{N_{p-2}}{N_{p-1}}}=Q \quad (12)$$

where N_P, N_{P-1}, N_{P-2} and G_P, G_{P-1}, G_{P-2} are the degrees of freedom and energy release rate with respect p, p-1, p-2 respectively. From Eq.(12), the exact energy release rate G_{ex} has been estimated.

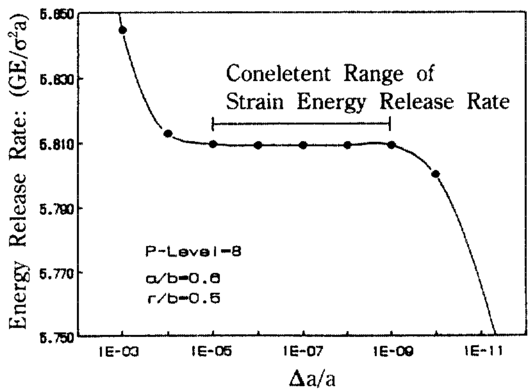

Fig. 2. Sensitivity of G vs. Δa in Prob. 3

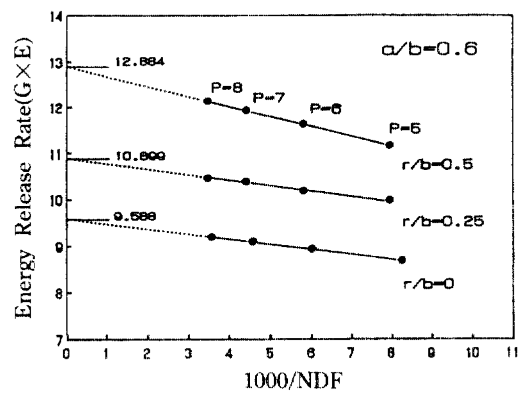

Fig. 3. Strong Linearity of $(NDF)^{-1}$ vs. G in Prob. 3

5. Numerical Results

5.1 Problem 1 : Membrane Behavior

Let us consider the well-known problem of a circular hole in a rectangular panel subjected to uniaxial tension. Because of the transfinite mapping to represent the circular boundary, the minimal meshes are needed. A typical h-version mesh by NASTRAN software and a p-version mesh in this study are shown in Fig. 4. Of interest is the maximum stress. Firstly, the variation of the stress concentration factor(K-factor) is investigated with respect to L/b ratio. It is known that K-factor is independent to L/b ratio since K=2.829 when

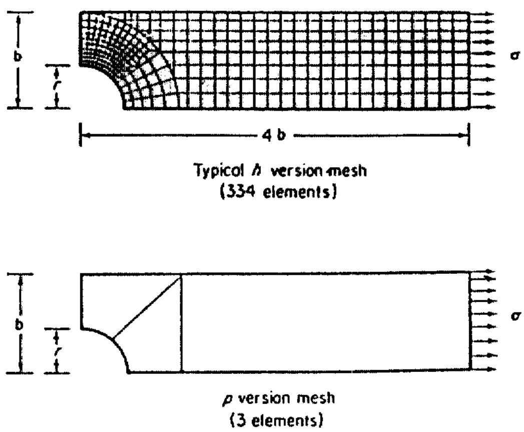

Fig. 4. P-version model of finite strips with a circular hole under tension

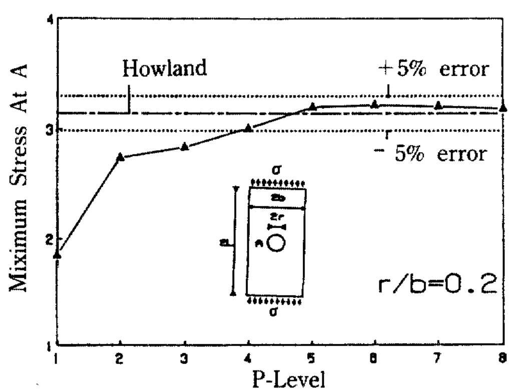

Fig. 5. Maximum stress at the point A

L/b=3, K=2.822 when L/b=4 and K=2.822 when L/b=6. In this study, p-version model with L/b=6 is fixed to compare with the empirical solutions. Secondly, the maximum stresses at the point A are computed corresponding to the different polynomial orders when r/b=0.2 shown in Fig. 5. The maximum stress of the point A obtained by p-version is bounded by 5% relative error in comparison with Howland's[9] when p-level exceeds p=4. It is known that the relative error in energy norm is under 1 % shown in Table 1 and that there is a strong linearity between Log N and Log $\|e\|_E$.

Using the same three-element mesh with p=8, the stress concentration factors(K) were computed

Table 1. Relative Error in Energy Norm

	Membrane Behavior		Flexural Behavior	
P-level	NDF	$\|e\|_E$(%)	NDF	$\|e\|_E$(%)
4	68	2.88	109	9.19
5	94	1.89	154	5.97
6	132	1.27	208	4.31
7	173	0.84	271	3.02
8	220	0.58	343	2.19

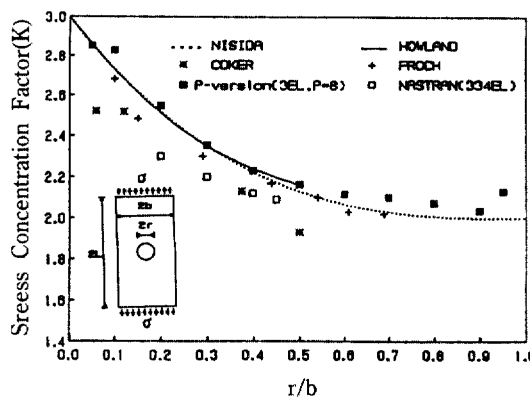

Fig. 6. Stress concentration factor with respect to r/b ratio

for a widely range of r/b ratios. The p-version models give an excellent agreement with the solutions of Howland which are valid for the range of r/b=0.1 to 0.5 and the experimental results by Nisida, Froch et al.[9] based on photoelastic approaches, which are shown in Fig.6. However, the NASTRAN 334-element model fails to get an acceptable solution. The stress concentration factor (K) is defined by.

$$K = \frac{\sigma_{max}}{\sigma_{nom}} \tag{13}$$

where σ_{nom} is nominal stress in the narrowest section such as :

$$\sigma_{nom} = \frac{P}{2(b-r)\cdot t} \tag{14}$$

P and t represent the applied axial force and thickness of section. Also, b and r are defined in Fig. 4. In general, K is close to 3.0 as r/b ratio goes to zero and when r/b = 1, K = 2.0. Nisida[9] proposed the following formula which can be valid between r/b=0.2 and 0.8.

$$K=2+\left(\frac{b-r}{b}\right)^3 \tag{15}$$

5.2 Problem 2 : Flexural Behavior

The rectangular panels with a circular hole subjected uniformly distributed bending moments are shown in Fig. 7. The stress concentration is influenced by both r/b and 2r/t. Reissner presented the solution of infinite plates(r/b=0) that includes the transverse shear deformation effects with respect to thickness(2r/t ratio). On the basis of Reissner solution, Nisida proposed the approximation of stress concentration factors that can be applied to the finite plate through the photoelastic experiments that follows;

$$K=1+\frac{1}{2}(K_{r/b=0}-1)\left\{1+\left(\frac{b-r}{b}\right)^3\right\} \tag{16}$$

where $K_{r/b=0}$ represents the stress concentration factors by Reissner. In this paper, the solutions of 3-element p-version model with p=8 are compared with those of Nisida. The stress concentration factor is calculated by Eq.(13), and σ_{nom} is defined by;

$$\sigma_{nom}=\frac{3M}{(b-r)\cdot t^2} \tag{17}$$

The stress concentration factors(K) with respect to 2r/t are plotted in Fig. 8. The p-version model with p=8 is close to Nisida's approximation when thickness is relatively large, however, the p-version solutions begin to show some amount of numerical errors as thickness is decreased.

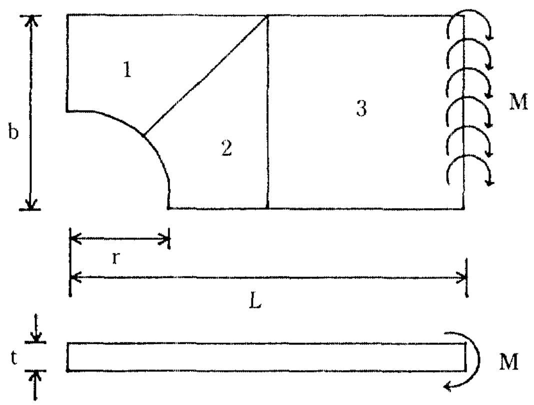

Fig. 7. P-version model of finite strips with circular hole under uniform bending

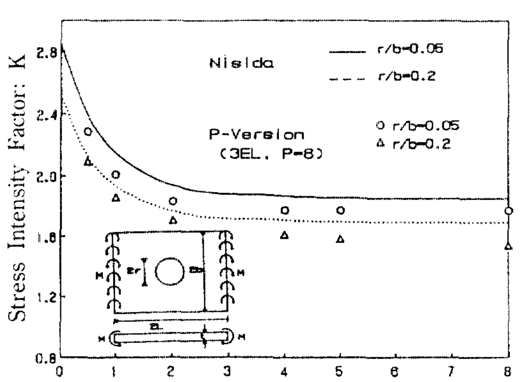

Fig. 8. Stress concentration factor with respect to 2r/t ratio

5.3 Problem 3 : Fracture Behavior

The p-version model for a crack emanating from a circular hole in rectangular plate under tension is shown in Fig.9. The stress intensity factor K_I

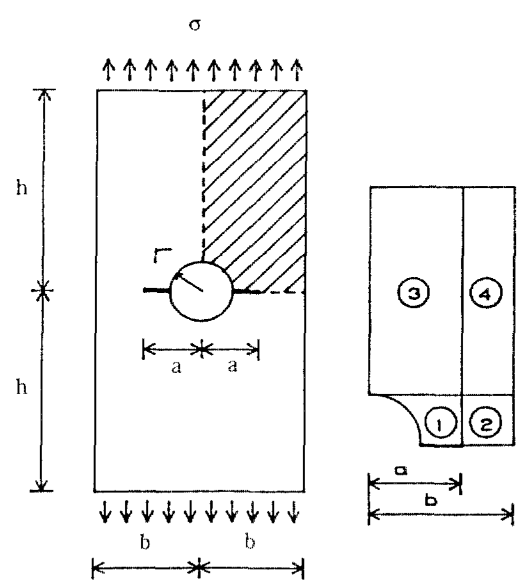

Fig. 9. P-version model for a crack emanating from a circular hole in rectangular plate under tension

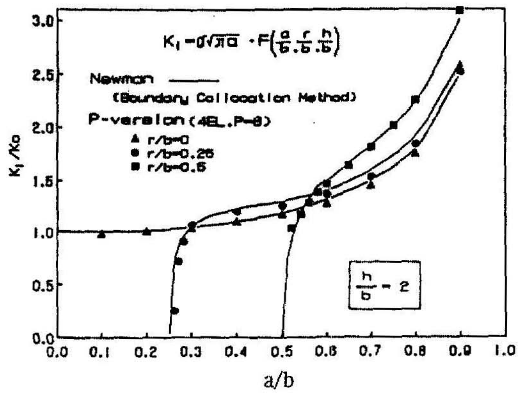

Fig. 10. Variation of stress intensity factor K_I with crack length

Table 2. Convergence Characteristics of F-factor when r/b=0

P-level	NDF	F-factor
1	13	0.75333
2	34	0.95754
3	55	0.98271
4	84	1.00930
5	121	1.03077
6	166	1.04230
7	219	1.04973
8	280	1.05475
∞	∞	1.075

is a function of a/b, r/b, and h/b where a is a half crack length; b is a half width of plate; r is a radius of circular hole; and h is a half length of plate. The influence of these parameters have been studied. Newman[10] carried out an assesment of the exact field based on boundary collocation method. It is seen that 4-element p-version model (p=8, h/b=2.0) gives good agreements with Newman's corresponding to r/b=0.0, 0.25 and 0.5 respectively. Due to the strong linearity, one can obtain the estimated exact value by extrapolation formula of Eq.(12) that is shown Fig. 3. The stress intensity factors are calculated by virtual crack extension method. The convergence characteristics of correction factors(F-factors) as p-level is increased are shown in Table 2 when r/b=0.0 which means centrally cracked panel problem. The F-factor with NDF=∞ was obtained by extrapolation method based on Eq.(12).

6. Conclusions

The solutions of 3-element p-version model with p=8 based on transfinite mapping technique are found to be successful in the case of membrane behavior problem for the wide ranges of r/b ratios. It is concluded that 3-element p-version model(p=8) has an excellent comparison with results by Nisida, Howland etc. However, the solutions of flexural behavior problem with same p-version model are smaller than those of Nisida under 10 % relative error as the ratio of r/b and 2r/t varies. In the fracture behavior problem under axial tension, 4-element p-version model(p=8, h/b=2.0) gives an excellent agreement with Newman's. As a result, p-version of the finite element computations using Legendre' polynomial is suitable for modeling the problems of stress concentration. All solutions are obtained by research purpose program P-FAP(P-version Fracture Analysis Program).

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