1. Introduction

In the general theory of relativity, the space-time universe is considered as a collection of all events which has a four dimensional differentiable manifold structure. In the theory of singularities, black holes, etc. causal structure of the space-time plays the important roles. Moreover, the uncertainties involved in the measurement which is implied by the quantum uncertainty principle are taken into account by the causality structures—Woodhause causality axiom [13].

The various causality structures have been developed and investigated their relations as well [1, 3, 5, 7, 11, 12, 13] in the recent developments concerning the relations among the various causality structure. Levichev [7] partially showed their relations for the homogeneous case.

As a model of the space-time, the homogeneous Lorentzian manifold is very restricted one, but it is physically and mathematically effective up to Lie group manifolds and Lie algebra.

In this paper we take a space-time as the homogeneous 4-dimensional Lorentzian manifold with the signature \((- + + +)\), and show more detailed relations of the causal structures and some properties of the causalities.

2. Preliminaries and main results

The Lorentzian manifold \(M\) is a paracompact Hausdorff manifold with one negative signature, and the homogeneous space-time \(M\) is a 4-dimensional Lorentzian manifold such that the group \(G\) of all isometric motions of \(M\) is transitive on it. We assume that \(M\) is connected, non-compact, and \(G\) acts leftly on \(M\). It clearly preserves time orientation. The notations one;

Partially supported by KOSEF.
Let us fix the point 0 in $M$ and $f(0) = x$ for $f$ in $G$. Then

$$f(I_0^+) = I_x^+, f(I_0^-) = I_x^-, f(J_0^+) = J_x^+, \text{ and } f(J_0^-) = J_x^- [7].$$

If $G$ is a simply transitive subgroup of the group on $M$, we may identify it with $M$ and the point 0 with the identity of the Lie group $M$. Levichev [7] showed that $J_0^+, J_0^-, I_0^+ \text{ and } I_0^-$ are subsemigroup ($I_0^+$ and $I_0^-$ may not contain the identity), $J_0^- = (J_0^+)^-$, $I_0^- = (I_0^+)^-$, $J_x^+ = x \cdot J_0^+$, $J_x^- = x \cdot J_0^-$, $I_x^+ = x \cdot I_0^+$ and $I_x^- = x \cdot I_0^-$ for all $x$ in $M$.

We also know that $A_0^+ = \uparrow I_0^-$ and $A_0^- = \downarrow I_0^+$ [1].

Thus, by the above properties, the following can be easily proved:

**Proposition 1.**

$$f(A_0^+) = A_x^+, f(A_0^-) = A_x^-.$$

$$f(I_0^+) = I_x^+, f(I_0^-) = I_x^-, f(J_0^+) = J_x^+, f(J_0^-) = J_x^-$$

$$f(\uparrow I_0^-) = \uparrow I_x^-, f(\downarrow I_0^+) = \downarrow I_x^+, f(J_0^+(0)) = J_x^+(x), f(J_0^-(0)) = J_x^-(x)$$

for all $x$ in $M$ and $f$ in $G$.

As like the uncertainty principle, the fact that a physical experiment can never be made sure to have performed at a precise event of the space-time, it leads to think of an event as the limit of a converging sequence of neighborhoods.

Thus Woodhouse [13] gives an axiom;

an event $x$ almost causally precedes another event $y$, denoted by $xAy$, if for all $x \in I_x^-$, $I_y^+ \subset I_x^+$; or equivalently, if for all $z \in I_y^+, I_x^- \subset I_z^-$. Thus axiom can be able to analyze reasonably causal influence under the uncertainty principle.

Let $A_x^+ = \{y \in M \mid xAy\}$ and $A_x^- = \{y \in M \mid yAx\}$. Woodhouse gives the principle of causality for a space-time such that if for any $x, y$ in $M$, $xAy$ and $yAx$, then it must be $x = y$. 

$$I_x^+(I_x^-); \text{ chronological future(past) set at } x$$

$$J_x^+(J_x^-); \text{ casual future(past) set at } x$$

$$\uparrow I_x^-(\downarrow I_x^+); \text{ chronological common future(past) set of } I_x^+(I_x^-)$$

$$J_s^+(x)(J_s^-(x)); \text{ Seifert casual future(past) set at } x$$

$$A_x^+(A_x^-); \text{ almost casual future(past) set at } x$$

$$\overline{A}(A^o); \text{ closure(interior) of } A$$

The definitions and the terminologies are referred to [2, 4, 8, 9, 10].
We say it Woodhause causality. Thus we have following property under our space-time.

**PROPOSITION 2.** $x \cdot A^+_0 = A^+_x$ for all $x$ in $M$.

*Proof.* Let $xm \in x \cdot A^+_0$ and $m \in A^+_0$. Then, for all $q \in I^+_q$, $I^+_m \subset I^+_q$. That is, every neighborhood $N_0$ of $0$ contains events which chronologically precedes some events in any neighborhood $N_m$ of $m$.

Pick an event $p$ in $I^+_m$ and $q$ in $I^+_0$, and let chronological future curves $\alpha, \beta$ such that $\alpha(0) = m, \alpha(1) = p, \beta(0) = q$ and $\beta(1) = p = \alpha(1)$. Consider chronological future curves $\gamma(t) = x\alpha(t)$ and $\eta(t) = x\beta(t)$. Then

$$\gamma(0) = x\alpha(0) = xm, \quad \gamma(1) = x\alpha(1) = xp,$$
$$\eta(0) = x\beta(0) = xq, \quad \eta(1) = x\beta(1) = xp = \gamma(1),$$
and $xq \in xI^+_0 = I^+_x$.

This shows $xm \in A^+_x$. That is, $xA^+_0 \subset A^+_x$.

Conversely, let $m \in A^+_x$ and $q \in I^+_0$. Since $I^+_x = xI^+_0, q = xq_1$, for some $q_1$ in $I^+_0$.

Thus there exist chronological future (timelike geodesic) curves $\alpha, \beta$ such that $\alpha(0) = m, \alpha(1) = p, \beta(0) = q = xq_1$, and $\beta(1) = p = \alpha(1)$.

If $s$ is an event in $A^+_0$, then for $q_1$ in $I^+_0$ and $p_1$ in $I^+_x$, there exist chronological future (timelike geodesic) curves $\alpha_1, \beta_1$, such that

$$\alpha_1(0) = s, \alpha_1(1) = p_1, \beta_1(0) = q_1 \text{ and } \beta_1(1) = p_1 = \alpha_1(1).$$

Let $\gamma(t) = x\alpha_1(t)$, and $\delta(t) = x\beta_1(t)$. Thus by the uniqueness of timelike geodesic from $xq_1$, $\beta = \delta$. Therefore $\beta(1) = \alpha(1)$. That is, $p = \beta(1) = \delta(1) = xp_1$ and $\alpha(1) = p = xp_1 = \delta(1) = \alpha_1(1) = \gamma(1) \in I^+_m$. Since $I^+_m$ is open and $\gamma(1) = \alpha(1) \in I^+_m$, $\alpha = \gamma$ by the same reason. Thus $m = \alpha(0) = \gamma(0) = xs \in xA^+_0$. This shows $A^+_x \subset xA^+_0$. The proof is complete.

The causalities of space-time were developed in the various ways and the relations of there were partially showed in many situation [1, 3, 7, 11, 12, 13] as follows;

(1) globally hyperbolic $\implies$ (2) causally simple $\implies$ (3) causally continuous $\implies$ (4) stably causal $\implies$ (5) woodhause causal $\implies$ (6)
strongly causal \implies (7) distinguishing \implies (8) future distinguishing
\implies (9) past distinguishing \implies (10) causal \implies (11) chronological \implies (12) non-compact.

The converse relations of these also have partially been showed under
the reflectingness or other cases [1, 5, 7, 11].

The followings were showed in [1, 7].

**Proposition A.** If a homogeneous space-time is future(past) distinguishing then it is past(future) distinguishing.

**Proposition B.** If a space-time is reflecting, then (4)-(9) are equivalent.

**Proposition C.** In a homogeneous space-time (3)-(9) are equivalent.

We say that \( J^+_K \) and \( J^-_K \) are causal future and past of the set \( K \) [4].

**Proposition 3.** If \( M \) is a homogeneous space-time, and \( J^+_0 (J^-_0) \) is closed, then \( M \) is causally simple.

**Proof.** Let \( K \) be a compact subset of \( M \). Then, since \( M \) is homogeneous, \( K J^+_0 \) is closed in \( M \).

By proposition 1,

\[
K J^+_0 = \{ k J^+_0 \mid k \in K \} = \{ J^+_k \mid k \in K \} = \bigcup_{k \in K} J^+_k = J^+_K.
\]

Similarly, \( J^-_K \) is closed. This completes the proof.

**Proposition 4.** If a null geodesically complete or naturally reductive, homogeneous space-time satisfies the generic condition, and the nonnegative Ricci tensor for all null vectors, then (3)-(11) are equivalent.

**Proof.** It can be proved from Proposition A,B,C, and Proposition 6.4.6 [4]. In the case of a naturally reductive Homogeneous space-time, the space-time is complete. Thus it can be showed by the same method.

Let \( F \) be a function which assigns to each \( x \) in \( M \) an open set \( F(x) \) in \( M \). \( F \) is called inner continuous if for any \( x \) and compact set \( K \subset F(x) \),
there is a neighborhood \( U \) of \( x \) such that \( K \subset F(z) \) for all \( z \) in \( U \). \( F \)
is outer continuous if for any \( x \) and any compact set \( K \subset M - F(x) \),
there is a neighborhood \( U \) of \( x \) such that \( K \subset M - F(z) \) for all \( z \) in \( U \).
DEFINITION. Let $F$ be a function which assigns to each $x$ in $M$ on closed set $F(x)$ in $M$. $F$ is $C$-outer continuous if for any $x$ and any compact set $K \subset M - F(x)$, there is a neighborhood $U$ of $x$ such that $K \subset M - F(z)$ for all $z$ in $U$.

Clearly, there is no relation between these kinds of continuities in general.

The several causal sets as set-function have partially been showed the inner or outer continuity of the sets under the different situations [5, 6].

Our space-time $M$ may be assumed homogeneous one.

PROPOSITION 5. $\uparrow I^-(\downarrow I^+)$ is inner continuous.

Proof. It suffices to show the case of $\uparrow I^-$ at 0 by Proposition 1. We know that $\uparrow I^-$ is open. Suppose $K$ is a compact subset of $\uparrow I^-$. Since $K$ is closed, there exists an open subset $V$ such that $K \subset V \subset \uparrow I_0^-$. Let $U = \{f \in G \mid f(K) \subset V\}$. Then $U$ is an open neighborhood of the identity of $G$, and may be $U = U^{-1}$. Thus $f(K) \subset \uparrow I_0^-$ for all $f$ in $U$. $V \subset f(\uparrow I_0^-)$ for all $f$ in $U$, and $K \subset f(\uparrow I_0^-)$ for all $f$ in $U$.

Let $W = \{f(0) \mid f \in U\}$. Then $W$ is an open neighborhood of 0, and $K \subset \uparrow I_x^-$ for all $x$ in $W$ by Proposition 1 and the above. This shows that $\uparrow I^-$ is continuous. Similarly $\downarrow I^+$ is inner continuous.

PROPOSITION 6. $A^+(A^-)$ is $C$-outer continuous.

Proof. It suffices to show the case of $A^+$ at 0 by Proposition 1. We know that $A_0^+$ is closed. Let $K$ be a compact subset of $M - A^+_0$. Then there exists an open subset $U$ such that $K \subset U \subset M - A^+_0$.

By the homogeneity of $M$, there exists an open subset $V$ of $G$ such that $f(K) \subset U$ for all $f$ in $V$, and $V$ may be equal to $V^{-1}$. Clearly $V$ is a neighborhood of the identity of $G$.

Thus, $K \subset M - \cup_{f \in V} f(A)$ if and only if $\cup_{f \in V} f(K) \subset M - A$.

Let $W = \{f(0) \mid f \in V\}$. Then $W$ is an open subset of $M$ containing 0, and $K \subset M - A_0^+$ for each $x$ in $W$. This shows that $A^+$ is $C$-outer continuous. Similarly $A^-$ is $C$-outer continuous.

PROPOSITION 7. $I^+(I^-)$ is outer continuous.

Proof. It can similarly be proved.
References


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