THE SUBHARMONIC BIFURCATION
IN AREA-PRESERVING MAPS

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1. Introduction

Many authors [1, 2, 3] have studied two-dimensional area-preserving maps as typical discrete versions of nonintegrable autonomous Hamiltonian systems with two degrees of freedom or nonautonomous systems with one degree of freedom. In particular, Van der Weele [3] has performed a bifurcation analysis based on Meyer [4] to prove the appearance of $n$-cycles from the elliptic fixed point at each resonance values.

The purpose of this paper is to present mathematically more clear and general methods to analyze the pattern of $n$-cycles bifurcating from the origin by means of the theory of normal forms [5, 6, 7, 8, 9] and the Liapunov-Schmidt method [10, 11].

Our bifurcation analysis can be compared with that of Hopf-bifurcation [7], however, the assumptions on the linear part of a map are quite different from each other. In the case of Hopf bifurcation, the complex conjugate eigenvalues of the linear part of a map cross the unit circle transversally as a parameter varies through 0, whereas in our case those eigenvalues move along the unit circle due to the area-preserving property of the map.

The main point of our analysis is that even if the normal form of an area-preserving map may not be area-preserving, the orbits, especially the $n$-cycles of the area-preserving map are locally diffeomorphic to those of the normal form, because the nonlinear change of coordinates leading to the normal form is a $\mu$-dependent local diffeomorphism.
2. The Normal form of an area-preserving map

Consider a two-dimensional area-preserving map $\mathbb{R}^2 \to \mathbb{R}^2$ of the following form [12, 13]

\begin{equation}
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
2c & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x \\
y
\end{bmatrix} + \begin{bmatrix}
f(x)
\end{bmatrix},
\end{equation}

where $f(x) = \sum_{k=2}^{\infty} a_k x^k$ is of class $C^\infty$ and $c$ is a real parameter.

For any $c \in \mathbb{R}$, the origin is a fixed point of (2.1) and its stability is determined by the eigenvalues

$$\lambda_{\pm} = c \pm \sqrt{c^2 - 1}$$

of the linear part. Note that $\lambda_+ \cdot \lambda_- = 1$, in agreement with the area preserving condition of (2.1). For $|c| \leq 1$, the eigenvalues lie on the unit circle, complex conjugate to each other and the origin becomes an elliptic fixed point.

Introducing a new parameter $\mu \in \mathbb{R}$ by writing

$$c = c(\mu) = \cos 2\pi(\theta_0 + \mu),$$

we have

\begin{equation}
(2.2)
\end{equation}

where $\theta_0 = \frac{m}{n}$ with $m$ and $n$ relatively prime integers. Then, we can rewrite (2.1) in the form

\begin{equation}
(2.3)
\end{equation}

where

$$A_\mu = D(x,y) G_\mu(0,0) = \begin{bmatrix}
2 \cos 2\pi(\theta_0 + \mu) & -1 \\
1 & 0
\end{bmatrix} \in \mathbb{R}^{2 \times 2}.$$

Let $\lambda(\mu)$ and $\bar{\lambda}(\mu)$ be the eigenvalues of $A_\mu$ and let $\lambda_0 = \lambda(0)$. Then we have

\begin{equation}
(2.4)
\end{equation}

$$\lambda(\mu) = e^{2\pi i(\theta_0 + \mu)} = \lambda_0 e^{2\pi i \mu}, \quad \lambda_0 = e^{2\pi i \theta_0}.$$
Notice that the eigenvalues move along the unit circle as $\mu$ varies through 0.

Now, we make a linear change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad \text{where}$$

$$P = \begin{bmatrix} 0 & 1 \\ -\sin 2\pi(\theta_0 + \mu) & \cos 2\pi(\theta_0 + \mu) \end{bmatrix},$$

to put (2.3) in the standard form,

$$\begin{bmatrix} \xi' \\ \eta' \end{bmatrix} = F_{\mu}(\xi, \eta) \equiv \begin{bmatrix} \cos 2\pi(\theta_0 + \mu) & -\sin 2\pi(\theta_0 + \mu) \\ \sin 2\pi(\theta_0 + \mu) & \cos 2\pi(\theta_0 + \mu) \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + f(\eta) \begin{bmatrix} \cot 2\pi(\theta_0 + \mu) \\ 1 \end{bmatrix},$$

where $F_{\mu} = P^{-1} \cdot G_{\mu} \cdot P$.

And again, by setting $z = \xi + i\eta$ and $\bar{z} = \xi - i\eta$ in (2.6), we can obtain the two-dimensional area-preserving map in complex form

$$(2.7) \quad z' = F_{\mu}(z) = \lambda(\mu)z + \frac{\lambda(\mu)}{i m \lambda(\mu)} \cdot f \left( \frac{z - \bar{z}}{2i} \right), \quad F_{\mu} : \mathbb{C} \rightarrow \mathbb{C},$$

where $\lambda(\mu) = \lambda_0 e^{2\pi i \mu} = \lambda_0 (1 + 2\pi i \mu + O(|\mu|^2))$, and

$$(2.8) \quad f \left( \frac{z - \bar{z}}{2i} \right) = \sum_{k=2}^{\infty} a_k \left( \frac{z - \bar{z}}{2i} \right)^k.$$

Let us write (2.7) in the form

$$(2.9) \quad z' = F_{\mu}(z) = \lambda(\mu)z + R(\mu, z, \bar{z}),$$

where $R(\mu, z, \bar{z}) = R_1(\mu, z, \bar{z}) + R_2(\mu, z, \bar{z}) + \ldots$ with $R_l(\mu, z, \bar{z}) = \sum_{p+q=l} c_{pq}(\mu) z^p \bar{z}^q, l \geq 2$. Then, from (2.8), the coefficients $c_{pq}(\mu)$ are given by

$$(2.10) \quad c_{20}(\mu) = -\frac{a_2}{4} m(\mu), \quad c_{11}(\mu) = \frac{a_2}{2} m(\mu), \quad c_{02}(\mu) = -\frac{a_2}{4} m(\mu),$$

$$c_{30}(\mu) = \frac{i a_3}{8} m(\mu), \quad c_{21}(\mu) = -\frac{3 i a_3}{8} m(\mu), \quad c_{12}(\mu) = \frac{3 i a_3}{8} m(\mu),$$

$$c_{03}(\mu) = -\frac{i a_3}{8} m(\mu), \ldots$$
with \( m(\mu) = \lambda(\mu)/\text{Im } \lambda(\mu). \)

Finally, we put (2.9) in a normal form by means of a \( \mu \)-dependent change of coordinates of the following form

(2.11) \[ z = w + \psi(\mu, w, \bar{w}) \equiv T_\mu(w), \]

where

\[
\psi(\mu, w, \bar{w}) = \psi_2(\mu, w, \bar{w}) + \psi_3(\mu, w, \bar{w}) + \ldots, \quad \text{and } \psi_1(\mu, w, \bar{w}) = \sum_{p+q=l} \psi_{pq}(\mu)w^p\bar{w}^q, \quad l \geq 2,
\]

with a suitable choice of the coefficients \( \psi_{pq}(\mu) \).

Then the new map in \( w \) becomes

\[ w' = \tilde{F}_\mu(w) = \left( T^{-1}_\mu \circ F_\mu \circ T_\mu \right)(w). \]

According to the theory of normal forms for maps [5, 6, 7, 8, 9], we can obtain the normal forms of \( F_\mu(z) \) as given in the following.

**Lemma 1.** Let \( F_\mu(z) = \lambda(\mu)z + R_2(\mu, z, \bar{z}) + R_3(\mu, z, \bar{z}) + \ldots \) with

\[
R_\ell(\mu, z, \bar{z}) = \sum_{p+q=\ell} c_{pq}(\mu)z^p\bar{z}^q, \quad \ell \geq 2 \quad \text{and} \quad \lambda(\mu) = \lambda_0 e^{2\pi i \theta_0}, \quad \text{where } \lambda_0 = e^{2\pi i \theta_0}.
\]

Then there exists a \( \mu \)-dependent local diffeomorphism of the form (2.11) which transforms the map \( F_\mu(z) \) to the following normal forms:

(i) when \( \theta_0 = \frac{1}{3} \)

\[
\tilde{F}_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + \eta_{21}(\mu)z^2\bar{z} + \eta_{40}(\mu)z^4 + \eta_{13}(\mu)z\bar{z}^3 + \mathcal{O}(|z|^5)
\]

If \( c_{02}(0) = 0 \), the term \( \bar{z}^2 \) can be removed in the normal form.

(ii) when \( \theta_0 = \frac{1}{4} \)

\[
\tilde{F}_\mu(z) = \lambda(\mu)z + \eta_{21}(\mu)z^2\bar{z} + \eta_{03}(\mu)\bar{z}^3 + \eta_{05}(\mu)z\bar{z}^5 + \eta_{14}(\mu)z\bar{z}^4 + \mathcal{O}(|z|^7)
\]

(iii) when \( \theta_0 = \frac{1}{n} \quad (n \geq 5) \)

\[
\tilde{F}_\mu(z) = \lambda(\mu)z + \eta_{21}(\mu)z^2\bar{z} + \eta_{0,n-1}(\mu)\bar{z}^{n-1} + \eta_{32}(\mu)z^{3}\bar{z}^2 + \mathcal{O}(|z|^7 + |z|^n)
\]
The coefficients \( \eta_{ij} \) can be calculated from those of \( F_\mu(z) \) as follows;

\[
\eta_{21}(0) = c_{21}(0) + \frac{|c_{11}(0)|^2}{1 - \lambda_0} + \frac{2|c_{02}(0)|^2}{\lambda_0^2 - \lambda_0} + \frac{2\lambda_0 - 1}{\lambda_0(1 - \lambda_0)} c_{11}(0) \cdot c_{20}(0)
\]

\[
\eta_{03}(0) = c_{03}(0) + \frac{c_{11}(0)c_{02}(0)}{\lambda_0^2 - \lambda_0} + \frac{2c_{02}(0) \cdot \tilde{c}_{20}(0)}{\lambda_0^2 - \lambda_0}.
\]

Furthermore, writing \( \tilde{F}_\mu(z) = \lambda(\mu)z + \tilde{R}(\mu, z, \bar{z}) \), \( \tilde{R}(\mu, z, \bar{z}) \) satisfies

\[
\tilde{R}(\mu, \lambda_0 z, \bar{\lambda}_0 \bar{z}) = \lambda_0 \tilde{R}(\mu, z, \bar{z})
\]

**Proof.** See [5], [6], [7].

As we mentioned in the introduction, the orbits of \( F_\mu(z) \) are locally diffeomorphic to those of \( \tilde{F}_\mu(z) \). Hence it is sufficient to examine the \( n \)-cycles of \( \tilde{F}_\mu(z) \). From now on, we write \( F_\mu(z) \) for \( \tilde{F}_\mu(z) \) for notational simplicity.

3. The Liapunov-Schimdt method [7]

Assume that \( \lambda_0^n = 1(\theta_0 = \frac{1}{n}) \) for \( n \geq 3 \). Let \( x = (x_1, \ldots, x_n) \in \mathbb{C}^n \) be a \( n \)-cycle of the map \( F_\mu(z) \), that is

\[
\begin{align*}
F_\mu(x_1) &= x_2 \\
F_\mu(x_2) &= x_3 \\
&\quad \vdots \\
F_\mu(x_n) &= x_1
\end{align*}
\]

(3.1)

where \( F_\mu(z) \) is in normal form as is given in Lemma 1. Let

\[
\mathcal{F}_\mu(x) = \begin{bmatrix} F_\mu(x_1) \\ \vdots \\ F_\mu(x_n) \end{bmatrix} = \begin{bmatrix} \lambda(\mu)x_1 + R(\mu, x_1, \bar{x}_1) \\ \vdots \\ \lambda(\mu)x_n + R(\mu, x_n, \bar{x}_n) \end{bmatrix}
\]

(3.2)

\[= \lambda(\mu)x + R(\mu, x, \bar{x}),\]

and

\[
S = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix}.
\]
Then (3.1) can be written as

\[(3.3) \quad Sx = F_\mu(x), \quad (F_\mu \in C^\infty(\mathbb{C}^n, \mathbb{C}^n), \quad S \in \mathbb{C}^{n \times n}).\]

To diagonalize $S$, we make a linear change of coordinates

\[(3.4) \quad y = Px, \quad y = (y_1, \ldots, y_n) \in \mathbb{C}^n,
\]

where

\[
P = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_0^{n-1} & \lambda_0^{2(n-1)} & \cdots & \lambda_0^{(n-1)(n-1)}
\end{bmatrix}.
\]

Then (3.3) is reduced to the equation

\[(3.5) \quad \Lambda y = P F_\mu(P^{-1}y),\]

where $\Lambda = \text{diag}(1, \lambda_0, \ldots, \lambda_0^{n-1})$.

If we define the map $\Phi: \mathbb{C}^n \times \mathbb{R} \rightarrow \mathbb{C}^n$ by

\[
\Phi(y, \mu) = P F_\mu(P^{-1}y) - \Lambda y = [\lambda(\mu)I - \Lambda]y + P R(\mu, P^{-1}y, \overline{P^{-1}y}),
\]

and let

\[(3.6) \quad L \equiv D_y \Phi(0, 0) = \lambda_0 I - \Lambda = \text{diag}(\lambda_0 - 1, \lambda_0 - \lambda_0, \ldots, \lambda_0 - \lambda_0^{n-1}),\]

then, since $\lambda_0 - \lambda_0^{n-1} = 0$, $L$ has rank $n - 1$, and $\ker L$ is a one dimensional subspace of $\mathbb{C}^n$ as

\[(3.7) \quad \ker L = \{y_n v_n | y_n \in \mathbb{C}, v_n = (0, \ldots, 0, 1)^T \in \mathbb{C}^n\}.
\]

By writing

\[
\mathbb{C}^n = \ker L \oplus (\ker L)^\perp,
\]

any $y \in \mathbb{C}^n$ can be written as

\[(3.8) \quad y = y_n v_n + w, \quad \text{where } v_n \in \ker L \text{ and } w \in (\ker L)^\perp = \text{Im } L.
\]
Let $E : \mathbb{C}^n \to (\ker L)^\perp$ be the projection. Then, $I - E : \mathbb{C}^n \to \ker L$, $Ey = (y_1, \ldots, y_{n-1}, 0)^T = w$, and $(I - E)y = y_nv_n$. Also, we can easily notice that $E, I - E$ and $L$ commute each other. Consequently, the equation $\Phi(y, \mu) = 0$ is equivalent to the following pair of equations

\begin{equation}
\begin{cases}
E\Phi(y_nv_n + w, \mu) = 0 & (a) \\
(I - E)\Phi(y_nv_n + w, \mu) = 0 & (b)
\end{cases}
\end{equation}

Notice that (3.10) (a) is uniquely solvable for $w$ as a function of $(y_n, \mu)$ near $(0, 0)$ by the implicit function theorem. Denoting $w = w^*(y_n, \mu)$, we can easily verify that [7]

\begin{equation}
\begin{cases}
w^*(y_n, \mu) = O(|\mu||y_n| + |y_n|^2) \\
w^*(\lambda_0 y_n, \mu) = \Lambda w^*(y_n, \mu)
\end{cases}
\end{equation}

After substituting $w^*(y_n, \mu)$ into (3.10) (b), we define a function $\gamma : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ by

\begin{equation}
\gamma(y_n, \mu) = \langle (I - E)\Phi(y_nv_n + w^*(y_n, \mu), \mu), v_n \rangle.
\end{equation}

Then, solutions $(y, \mu)$ of $\Phi(y, \mu) = 0$ are locally one-to-one correspondence to the solutions $(y_n, \mu)$ of $\gamma(y_n, \mu) = 0$ via the relation

\begin{equation}
y = y_nv_n + w^*(y_n, \mu).
\end{equation}

From (3.6), we have

\begin{equation}
\gamma(y_n, \mu) = \langle \Phi(y_nv_n + w^*(y_n, \mu), \mu), v_n \rangle \\
= (\lambda(\mu) - \lambda_0)y_n + \left\langle P\mathcal{R}(\mu, P^{-1}y, P^{-1}y), v_n \right\rangle,
\end{equation}

where $y$ is given in (3.13).

**Lemma 2.** Let $z = \frac{1}{n}y_n$. Then the equation $\gamma(y_n, \mu) = 0$ is equivalent to the following equation in $\mathbb{C}$:

\begin{equation}
\lambda_0 z = F_\mu(z) = \lambda(\mu)z + R(\mu, z, \bar{z}),
\end{equation}
where \( F_\mu(z) \) is in normal form.

Proof. Letting \( z = \frac{1}{n} y_n \) in (3.14), \( \gamma(y_n, \mu) = 0 \) becomes

\[
\lambda_0 z = \lambda(\mu)z + \frac{1}{n} \left\langle PR(\mu, P^{-1}y, P^{-1}y), v_n \right\rangle.
\]

Recall that \( F_\mu(z) = \lambda(\mu)z + R(\mu, z, \bar{z}) \), where \( R(\mu, z, \bar{z}) = \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}) \) is in normal form up to order \( r \) and hence \( \tilde{R}(\mu, \lambda_0 z, \bar{\lambda}_0 \bar{z}) = \lambda_0 \tilde{R}(\mu, z, \bar{z}) \). From (3.2) and (3.4) we have

\[
\left\langle PR(\mu, P^{-1}y, P^{-1}y), v_n \right\rangle = R(\mu, x_1, x_1) + \lambda_0 R(\mu, x_2, x_2) + \ldots + \lambda_0^{n-1} R(\mu, x_n, x_n),
\]

where \( \{x_1, \ldots, x_n\} \) is the \( n \)-cycle for the system (3.1) given by

\[
x_k = (P^{-1}y)_k = [P^{-1}(y_n v_n + w^*(y_n, \mu))]_k
\]

\[
= \frac{1}{n} \sum_{j=1}^{n-1} \lambda_0^{(k-1)(j-1)} w_j^*(y_n, \mu) + \frac{1}{n} \lambda_0^{(k-1)(n-1)} y_n
\]

\[
= \frac{1}{n} \sum_{j=1}^{n-1} \lambda_0^{(k-1)(j-1)} w_j^*(nz, \mu) + \lambda_0^{k-1} z \quad (k = 1, 2, \ldots, n).
\]

Note that if we write

\[
x_1 = \varphi_\mu(z) \equiv z + \frac{1}{n} \sum_{j=1}^{n-1} w_j^*(nz, \mu) = z + \mathcal{O}(|\mu||\bar{z}| + |\bar{z}|^2),
\]

then the other \( n \)-periodic points \( x_2, \ldots, x_n \) can be obtained from the property (3.11) as

\[
x_2 = \varphi_\mu(\lambda_0 z), x_3 = \varphi_\mu(\lambda_0^2 z), \ldots, x_n = \varphi_\mu(\lambda_0^{n-1} z).
\]

Then we have

\[
R(\mu, x_1, x_1) = R(\mu, \varphi_\mu(z), \varphi_\mu(z))
\]

\[
= \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1})
\]

\[
\bar{\lambda}_0 R(\mu, x_2, x_2) = \bar{\lambda}_0 R(\mu, \varphi_\mu(\lambda_0 z), \varphi_\mu(\lambda_0 z))
\]

\[
= \bar{\lambda}_0 R(\mu, \lambda_0 z + \mathcal{O}(|\mu||z| + |z|^2), \bar{\lambda}_0 \bar{z} + \mathcal{O}(|\mu||z| + |z|^2))
\]

\[
= \tilde{R}(\mu, z, \bar{z}) + \mathcal{O}(|z|^{r+1}).
\]
Similarly, \( \tilde{\lambda}_0^{n-1} R(\mu, x_n, \bar{x}_n) = \tilde{R}(\mu, z, \bar{z}) + O(|z|^{r+1}) \). Therefore

\[
\frac{1}{n} \left\langle P \mathcal{R}(\mu, P^{-1} y, \overline{P^{-1}} y), v_n \right\rangle = \tilde{R}(\mu, z, \bar{z}) + O(|z|^{r+1}).
\]

Consequently, \( g(y_n, \mu) = 0 \) is equivalent to

\[
\lambda_0 z = \lambda(\mu) z + \tilde{R}(\mu, z, \bar{z}) + O(|z|^{r+1}) = F_\mu(z)
\]

Thus, our problem to find the \( n \)-periodic fixed points for the area-preserving map \( F_\mu(z) \) written in the normal form has been reduced to solving the scalar equation (3.15) and the coordinates of the \( n \)-periodic fixed points are given by (3.16) and (3.17).


(i) The case \( n = 3 \) and \( a_2 \neq 0 \)

In this case, \( \lambda_0 = e^{2\pi i/3} \) and from the Lemma 1, we have,

\[
F_\mu(z) = \lambda(\mu) z + c_{02}(\mu) \bar{z}^2 + O(|z|^3),
\]

where \( \lambda(\mu) = \lambda_0(1 + \mu \lambda_1 + O(|\mu|^2)) \) with \( \lambda_1 = 2\pi i \) and

\[
c_{02}(0) = -\frac{a_2}{2\sqrt{3}} \lambda_0.
\]

Then (3.15) becomes

\[
\mu \lambda_1 z + \tilde{\lambda}_0 c_{02}(0) \bar{z}^2 + O(|\mu|^2 |z| + |\mu||z|^2 + |z|^3) = 0.
\]

Let \( z = re^{2\pi i \varphi} \). Then we have,

\[
\mu \lambda_1 r e^{2\pi i \varphi} + \tilde{\lambda}_0 c_{02}(0) r^2 e^{-4\pi i \varphi} + g_1(\mu, re^{2\pi i \varphi}, re^{-2\pi i \varphi}) = 0,
\]

where \( g_1(\mu, re^{2\pi i \varphi}, re^{-2\pi i \varphi}) = O(|\mu|^2 r + |\mu|r^2 + r^3) \).

Separating the trivial solution \( r = 0 \), we have

\[
\mu \lambda_1 + \tilde{\lambda}_0 c_{02}(0) r e^{-6\pi i \varphi} + g(\mu, r, \varphi) = 0,
\]
or

\begin{equation}
2\pi i \mu - \frac{a_2}{2\sqrt{3}} e^{-6\pi i \varphi} + g(\mu, r, \varphi) = 0,
\end{equation}

where

\[ g(\mu, r, \varphi) = r^{-1} e^{-2\pi i \varphi} g_1(\mu, r e^{2\pi i \varphi}, r e^{-2\pi i \varphi}) \]

\[ = \mathcal{O}(|\mu|^2 + |\mu| r + r^2). \]

Note that \( g(\mu, r, \varphi) \) has the following property

\[ g(\mu, r, \varphi + \frac{1}{3}) = g(\mu, r, \varphi). \]

Now, if we set

\begin{equation}
\begin{cases}
    r = 4\pi \sqrt{3} |\frac{\mu}{a_2}| (1 + r_1) \\
    \varphi = \varphi_0 + \varphi_1, \quad \varphi_0 = -\frac{1}{6\pi} \text{arg}(\frac{\mu}{a_2}) \pmod{\frac{1}{3}},
\end{cases}
\end{equation}

substituting (4.4) in (4.3) and simplifying (4.3), we have

\[ \mu - \mu e^{-6\pi i \varphi_1} (1 + r_1) + g(\mu, 4\pi \sqrt{3} |\frac{\mu}{a_2}| (1 + r_1), \varphi_0 + \varphi_1) = 0. \]

Set

\[ h(\mu, r_1, \varphi_1) = 1 - e^{-6\pi i \varphi_1} (1 + r_1) + g_2(\mu, r_1, \varphi_1), \]

where \( g_2(\mu, r_1, \varphi_1) = \mu^{-1} g(\mu, 4\pi \sqrt{3} |\frac{\mu}{a_2}| (1 + r_1), \varphi_0 + \varphi_1) = \mathcal{O}(|\mu|). \)

Since

\[ h(0, 0, 0) = 0, \quad \frac{\partial h}{\partial r_1}(0, 0, 0) = -1, \quad \frac{\partial h}{\partial \varphi_1}(0, 0, 0) = 6\pi i, \]

by the implicit function theorem, we have

\[ r_1 = r_1(\mu), \quad r_1(0) = 0, \quad \varphi_1 = \varphi_1(\mu), \quad \varphi_1(0) = 0. \]

Consequently, we have

\begin{equation}
\begin{cases}
    r = 4\pi \sqrt{3} |\frac{\mu}{a_2}| + \mathcal{O}(|\mu|^2) \\
    \varphi = -\frac{1}{6\pi} \text{arg}(\frac{\mu}{a_2}) + \mathcal{O}(|\mu|) \pmod{\frac{1}{3}}.
\end{cases}
\end{equation}
and the coordinate of the 3-periodic points for the area-preserving map $F_\mu(z)$ in normal form is given, from (3.16), (3.17), (4.5), by

$$\begin{align*}
x_1 &= \varphi_\mu(z) = z(\mu) + O(|\mu||z| + |z|^2) \\
&= r(\mu)e^{2\pi i \varphi(\mu)} + O(|\mu|^2) \\
&= 4\pi \sqrt{3} |\frac{\mu}{a_2}| e^{2\pi i \varphi_0} + O(|\mu|^2), \\
x_2 &= \varphi_\mu(\lambda_0 z) \\
x_3 &= \varphi_\mu(\lambda_0^2 z),
\end{align*}$$

(4.6)

where $\varphi_0 = -\frac{1}{6\pi} \text{arg}\left(\frac{\mu}{a_2}\right)$ and $\lambda_0 = e^{2\pi i/3}$.

Notice that as $\mu$ varies from $\mu < 0$ to $\mu > 0$, $\text{arg}\left(\frac{\mu}{a_2}\right)$ changes by $\pi$, and hence the orientation of the 3-cycle is reversed as $\mu$ crosses 0 (Fig. 1).

Also note that the 3-cycle of the original area-preserving map (2.9) is given in the same form as in (4.6), since, for $\mu$ near 0, the original map (2.9) is transformed to the normal forms via the near-identity transformation of the form (2.11).

To examine the stability of the 3-cycle for the map

$$F_\mu(z) = \lambda(\mu)z + c_{02}(\mu)\bar{z}^2 + O(|z|^3),$$

we consider the map

$$F_\mu^3(z) = [1 + 3\mu \lambda_1 + O(|\mu|^2)]z + 3c_{02}(0)\bar{\lambda}_0 \bar{z}^2 + O(|\mu||z|^2 + |z|^3).$$

Then, we can easily see that one of the eigenvalues of the Jacobian $\partial(F_\mu^3(z), F_\mu^3(z))/\partial(z, \bar{z})$ at one of the 3 fixed points in (4.6) is outside the unit circle and the other inside it, so the 3-cycle is hyperbolic(saddle) on both sides of $\mu = 0$.

We can summarize the above results in the following theorem.

**Theorem 1.** Let $F_\mu : \mathbb{C} \to \mathbb{C}$ be the map in complex form given in (2.7) and assume that $\lambda_0^3 = 1(\lambda_0 \neq 1)$ and $a_2 \neq 0$. Then, a one-parameter family of 3-periodic fixed points $\{(x_1(\mu), x_2(\mu), x_3(\mu))| \mu \in \mathbb{R}\}$ undergoes transcritical bifurcation from the origin (elliptic fixed
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point) and reverses the orientation as $\mu$ crosses 0. The 3-periodic points are given by

\begin{align*}
x_1(\mu) &= r(\mu)e^{2\pi i \varphi(\mu)} + \mathcal{O}(|\mu|^2) \\
x_2(\mu) &= r(\mu)e^{2\pi i (\varphi(\mu) + \frac{1}{3})} + \mathcal{O}(|\mu|^2) \\
x_3(\mu) &= r(\mu)e^{2\pi i (\varphi(\mu) + \frac{2}{3})} + \mathcal{O}(|\mu|^2),
\end{align*}

where

\begin{align*}
r(\mu) &= 4\pi \sqrt{3} \left| \frac{\mu}{\alpha_2} \right| + \mathcal{O}(|\mu|^2) \\
\varphi(\mu) &= -\frac{1}{6\pi} \arg \left( \frac{i\mu}{\alpha_2} \right) + \mathcal{O}(|\mu|) \pmod{\frac{1}{3}}
\end{align*}

and they are hyperbolic (saddle) on both sides of $\mu = 0$.

(ii) The case $n = 3$ and $\alpha_2 = 0$

In this case, $c_{pq}(\mu) = 0$ for all $p, q$ with $p+q = 2$ and from Lemma 1, we have the normal form,

\begin{equation}(4.7) \quad F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2 \bar{z} + \beta(\mu)z^4 + \gamma(\mu)z^2 \bar{z}^3 + \mathcal{O}(|z|^5),\end{equation}

where the coefficients $\alpha_0 \equiv \alpha(0)$, $\beta_0 \equiv \beta(0)$ and $\gamma_0 \equiv \gamma(0)$ are given by

\begin{equation}(4.8) \quad \begin{cases}
\alpha_0 = c_{21}(0) = -\frac{3ia_3}{8} \frac{\lambda_0}{\Im \lambda_0} = -\frac{\sqrt{3}i}{4} a_3 \lambda_0 \\
\beta_0 = c_{40}(0) = \frac{a_4 \lambda_0}{16 \Im \lambda_0} = \frac{a_4}{8\sqrt{3}} \lambda_0 \\
\gamma_0 = c_{13}(0) = -\frac{a_4 \lambda_0}{4 \Im \lambda_0} = -\frac{a_4}{2\sqrt{3}} \lambda_0
\end{cases}\end{equation}

Eq. (3.5) becomes

\begin{equation}(4.9) \quad \mu \lambda_1 z + \bar{\lambda}_0 \alpha_0 z^2 \bar{z} + \bar{\lambda}_0 \beta_0 z^4 + \bar{\lambda}_0 \gamma_0 \bar{z}^3 z \\
+ \mathcal{O}(|\mu|^2 |z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0.\end{equation}
Let \( z = r e^{2\pi i \varphi} \). Then,

\[
\begin{align*}
\mu \lambda_1 r e^{2\pi i \varphi} + \lambda_0 \alpha_0 r^3 e^{2\pi i \varphi} + \lambda_0 \beta_0 r^4 e^{8\pi i \varphi} + \lambda_0 \gamma_0 r^4 e^{-4\pi i \varphi} \\
+ \mathcal{O}(|\mu|^2 r + |\mu| r^3 + |\mu| r^4 + r^5) &= 0.
\end{align*}
\]

Separating the trivial solution \( r = 0 \),

\[
\begin{align*}
\mu \lambda_1 + \lambda_0 \alpha_0 r^2 + \lambda_0 \beta_0 r^4 e^{6\pi i \varphi} + \lambda_0 \gamma_0 r^4 e^{-6\pi i \varphi} \\
+ \mathcal{O}(|\mu|^2 + |\mu| r^2 + |\mu| r^3 + r^4) &= 0.
\end{align*}
\]

or

\[
(4.10) \quad 2\pi i \mu - \frac{\sqrt{3} i}{4} a_3 r^2 + \frac{a_4}{8\sqrt{3}} r^3 e^{6\pi i \varphi} - \frac{a_4}{2\sqrt{3}} r^3 e^{-6\pi i \varphi} \\
+ \mathcal{O}(|\mu|^2 + |\mu| r^2 + |\mu| r^3 + r^4) = 0.
\]

Set

\[
(4.11) \quad \begin{cases} 
\mu = \mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3, \\
\varphi = \varphi_0 + \varphi_1
\end{cases}
\]

where \( \mu_0, \mu_1, \mu_2, \varphi_0 \) and \( \varphi_1 \) are to be determined.

Substituting (4.11) into (4.10),

\[
2\pi i (\mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3) - \frac{\sqrt{3}}{4} i a_3 r^2 + \frac{a_4}{8\sqrt{3}} r^3 e^{6\pi i \varphi} \\
- \frac{a_4}{2\sqrt{3}} r^3 e^{-6\pi i \varphi} + \mathcal{O}(r^4) = 0.
\]

First, choose \( \mu_0 \) so that \( 2\pi i \mu_0 - \frac{\sqrt{3}}{4} i a_3 = 0 \), then

\[
(4.12) \quad \mu_0 = \frac{\sqrt{3}}{8\pi} a_3.
\]

With this choice of \( \mu_0 \), we have

\[
(4.13) \quad [2\pi i \mu_1 + \frac{a_4}{8\sqrt{3}} (e^{6\pi i \varphi} - 4 e^{-6\pi i \varphi})] r^3 + 2\pi i \mu_2 r^3 + \mathcal{O}(r^4) = 0.
\]
Next, we choose $\mu_1$ and $\varphi_0$ so that

$$2\pi i \mu_1 + \frac{a_4}{8\sqrt{3}}(e^{6\pi i \varphi_0} - 4e^{-6\pi i \varphi_0}) = 0.$$ 

Note that since $\mu_1$ and $a_4$ are real, $e^{6\pi i \varphi_0} - 4e^{-6\pi i \varphi_0}$ must be pure imaginary and this happens only when $e^{6\pi i \varphi_0}$ is pure imaginary, that is,

$$6\pi \varphi_0 = \pm \frac{\pi}{2} \pmod{2\pi}$$

or

(4.14) \hspace{1cm} \varphi_0^{(1),(2)} = \pm \frac{1}{12} \pmod{\frac{1}{3}}.

If $\varphi_0 = \varphi_0^{(1)} = \frac{1}{12}$, then

(4.15) \hspace{1cm} \mu_1 = \mu_1^{(1)} = -\frac{5a_4}{16\pi \sqrt{3}}.

If $\varphi_0 = \varphi_0^{(2)} = -\frac{1}{12}$, then

(4.16) \hspace{1cm} \mu_1 = \mu_1^{(2)} = \frac{5a_4}{16\pi \sqrt{3}}.

Now, from (4.13), we let

$$h(\mu_2, \varphi, r) = 2\pi i \mu_1 + \frac{a_4}{8\sqrt{3}}(e^{6\pi i \varphi} - 4e^{-6\pi i \varphi}) + 2\pi i \mu_2 + O(r).$$

Then,

$$h(0, \varphi_0, 0) = 0$$

$$\frac{\partial h}{\partial \mu_2}(0, \varphi_0, 0) = 2\pi i$$

$$\frac{\partial h}{\partial \varphi}(0, \varphi_0, 0) = \frac{a_4}{4\sqrt{3}} \cdot 3\pi i(e^{6\pi i \varphi_0} + 4e^{-6\pi i \varphi_0}) = \pm \frac{a_4}{4\sqrt{3}} 9\pi,$$
and by the implicit function theorem, we know that \( \mu_2 = \mu_2(r) \), \( \varphi = \varphi(r) \) and \( \mu_2(0) = 0 \), \( \varphi_1(0) = 0 \). Thus, we have a pair of 3-cycles
\[ z = re^{2\pi i \varphi(r)} \], where \( r \) is regarded as a parameter which is related to \( \mu \) as

\[
\begin{align*}
\mu^{(1)} &= \frac{\sqrt{3}}{8\pi} a_3 r^2 - \frac{5}{16\pi \sqrt{3}} a_4 r^3 + O(r^4) \\
\varphi^{(1)} &= \frac{1}{12} + O(r) \quad \text{(mod } \frac{1}{2}) \\
\mu^{(2)} &= \frac{\sqrt{3}}{8\pi} a_3 r^2 + \frac{5}{16\pi \sqrt{3}} a_4 r^3 + O(r^4) \\
\varphi^{(2)} &= -\frac{1}{12} + O(r) \quad \text{(mod } \frac{1}{2})
\end{align*}
\]

Note that if \( a_3 > 0 \), \( \mu \) must be greater than 0 and we have a supercritical bifurcation and if \( a_3 < 0 \), \( \mu \) must be less than 0 and have a subcritical bifurcation (Fig. 2).

To study the stability of the pair of 3-cycles for the map

\[ F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2 \bar{z} + \beta(\mu)z^4 + \gamma(\mu)z \bar{z}^3 + O(|z|^5), \]

we examine the eigenvalues of the map

\[
\begin{align*}
z' = F_\mu^3(z) &= (1 + 3\mu \lambda_1)z + 3\bar{\lambda}_0 a_0 z^2 \bar{z} \\
&\quad + 3\bar{\lambda}_0 \beta_0 z^4 + 3\bar{\lambda}_0 \gamma_0 z \bar{z}^3 \\
&\quad + O(|\mu|^2 |z| + |\mu||z|^3 + |\mu||z|^4 + |z|^5) = 0.
\end{align*}
\]

Let \( \sigma_1, \sigma_2 \) are the eigenvalues of the Jacobian \( A = \partial(z', \bar{z}')/\partial(z, \bar{z}) \) at one of the 3 fixed points \( x \) of one family for \( F_\mu^3(z) \). If we assume that we used an area-preserving transformation of the form (2.11), then we can easily check that if \( a_3 a_4 > 0 \), \( \sigma_1 \) and \( \sigma_2 \) are real and reciprocal for \( \mu = \mu^{(2)} \), and \( \sigma_1 \) and \( \sigma_2 \) are complex conjugate on the unit circle for \( \mu = \mu^{(1)} \). If \( a_3 a_4 < 0 \), the situation is reversed.

Therefore, we can state the following theorem.

**Theorem 2.** Let \( F_\mu : \mathbb{C} \to \mathbb{C} \) be the map given in (2.7) and assume that \( \lambda_0^3 = 1(\lambda_0 \neq 1) \) and \( a_2 = 0 \). Then a pair of two one-parameter families of 3-periodic fixed points

\[
\{(x_1^{(j)}(r), x_2^{(j)}(r), x_3^{(j)}(r)) | r \in \mathbb{R}^+ \} \quad (j = 1, 2)
\]
bifurcate from the origin on the same side of $\mu = 0$. If $a_3 > 0$, or $< 0$ we have a supercritical or subcritical bifurcation respectively. Those $3$-periodic points are given by

$$
\begin{align*}
x_1^{(j)}(r) &= x_1^{(j)}(r) = re^{2\pi i \varphi^{(j)}(r)} + O(r^2) \\
x_2^{(j)}(r) &= x_2^{(j)}(r) = re^{2\pi i (\varphi^{(j)}(r) + \frac{1}{3})} + O(r^2) \\
x_3^{(j)}(r) &= x_3^{(j)}(r) = re^{2\pi i (\varphi^{(j)}(r) + \frac{2}{3})} + O(r^2)
\end{align*}
$$

for $j = 1, 2$.

where $r$ is related to $\mu$ as in (4.17).

Moreover, those $3$-periodic points with smaller $r$ is hyperbolic (saddle) and those with larger $r$ is elliptic.

(iii) The case $n = 4$

Let $\lambda_0 = e^{2\pi i/4} = i$.

Then the normal form of $F_\mu(z)$ is

$$
F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2\bar{z} + \beta(\mu)\bar{z}^3 + O(|z|^5),
$$

where $\alpha(0) \equiv \alpha_0$ and $\beta(0) \equiv \beta_0$ are related to the coefficients of the original equation as follows

$$
\begin{align*}
\alpha_0 &= \frac{1}{8} (3a_3 + a_2^2) \\
\beta_0 &= \frac{1}{8} (a_3 - a_2^2).
\end{align*}
$$

Then eq. (3.15) becomes

$$
2\pi i \mu z + c_1 z^2\bar{z} + c_2 \bar{z}^3 + g_1(\mu, z, \bar{z}) = 0,
$$

where

$$
\begin{align*}
c_1 &= \bar{\lambda}_0 \alpha_0 = -\frac{i}{8} (3a_3 + a_2^2) \\
c_2 &= \bar{\lambda}_0 \beta_0 = \frac{i}{8} (a_2^2 - a_3) \\
g_1(\mu, z, \bar{z}) &= O(|\mu|^2|z| + |\mu||z|^3 + |z|^5).
\end{align*}
$$
Setting \( z = re^{2\pi i\varphi} \) and separating the trivial solution \( r = 0 \), we have

\[
2\pi i\mu + c_1 r^2 + c_2 r^2 e^{-8\pi i\varphi} + g(\mu, r, \varphi) = 0,
\]

where \( g(\mu, r, \varphi) = \mathcal{O}(|\mu|^2 + |\mu|r^2 + r^4) \).

To look for the principal part, put

\[
\begin{aligned}
\mu &= \mu_0 r^2 + \mu_1 r^2 \\
\varphi &= \varphi_0 + \varphi_1
\end{aligned}
\]

where \( \mu_0, \mu_1, \varphi_0 \) and \( \varphi_1 \) are to be determined.

Substituting (4.22) in (4.21) and dividing by \( r^2 \), we have

\[
2\pi i\mu_0 + c_1 + c_2 e^{-8\pi i\varphi} + 2\pi i\mu_1 + f_1(\mu_1, r, \varphi) = 0,
\]

where \( f_1(\mu_1, r, \varphi) = \mathcal{O}(r^2) \). We choose \( \mu_0 \) and \( \varphi_0 \) so that

\[
2\pi i\mu_0 + c_1 + c_2 e^{-8\pi i\varphi_0} = 0.
\]

If \( c_2 \neq 0 \), i.e., \( a_3 \neq a_2^2 \), we have

\[
e^{-8\pi i\varphi_0} = \frac{2\pi i\mu_0 + c_1}{c_2} = \frac{a_2^2 + 3a_3 - 16\pi \mu_0}{a_2^2 - a_3}.
\]

Since \( e^{-8\pi i\varphi_0} \) must be real, we must have

\[
(4.24) \quad \varphi_0 = \varphi_0^{(1)} = 0 \quad \text{or} \quad \varphi_0 = \varphi_0^{(2)} = \frac{1}{8} \quad (\text{mod} \frac{1}{4}),
\]

and for each value of \( \varphi_0, \mu_0 \) can be determined as

\[
(4.25) \quad \mu_0 = \mu_0^{(1)} = \frac{a_3}{4\pi} \quad \text{for} \ \varphi_0^{(1)} \quad \text{and} \quad \mu_0 = \mu_0^{(2)} = \frac{a_3 + a_2^2}{8\pi} \quad \text{for} \ \varphi_0^{(2)}.
\]

If \( c_2 = 0 \), i.e., \( a_3 = a_2^2 \), we have one solution for \( \mu_0 \)

\[
(4.26) \quad \mu_0 = -\frac{c_1}{2\pi i} = \frac{3a_3 + a_2^2}{16\pi}.
\]
However, this is not the generic case. Furthermore, if we define $h(J-L, r, \varphi)$ as following

$$h(J-L, r, \varphi) = (2\pi i \mu_0 + c_1 + c_2 e^{-8\pi i \varphi}) + 2\pi i \mu_1 + f_1(\mu_1, r_1, \varphi),$$

we have

$$h(0, 0, \varphi_0) = 0$$

$$\frac{\partial h}{\partial \mu_1}(0, 0, \varphi_0) = 2\pi i$$

$$\frac{\partial h}{\partial \varphi}(0, 0, \varphi_0) = -8\pi ic_2 e^{-8\pi i \varphi_0}$$

$$= \pm \pi (a_2^2 - a_3) \quad (\pm \text{according as } \varphi_0 = \varphi_0^{(1)} \text{ or } \varphi_0^{(2)}),$$

and hence the implicit function theorem is applicable only if $a_3 \neq a_2$. Thus, in this generic case, from the evenness of $f_1(\mu_1, r, \varphi)$ in $r$, we have

$$\mu_1 = \mu_1(r) = \mathcal{O}(r^2), \quad \varphi_1 = \varphi_1(r) = \mathcal{O}(r^2).$$

Therefore, generically we have two one-parameter families of 4-cycles, $z = z^{(j)}(r) = r e^{2\pi i \varphi^{(j)}(r)} (j = 1, 2)$, bifurcating from the origin, and the parameters $\mu$ and $r$ are given as

$$\begin{cases} 
\mu^{(j)} = \mu_0^{(j)} r^2 + \mathcal{O}(r^4) \\
\varphi^{(j)} = \varphi_0^{(j)} + \mathcal{O}(r^2)
\end{cases}$$

(4.27)

where $\mu_0^{(j)}$ and $\varphi_0^{(j)}$ $(j = 1, 2)$ are given in (4.24) and (4.25). Notice from (4.25) that if $a_3 > 0$ or $a_3 < -a_2^2$, then $\mu_0^{(1)} \mu_0^{(2)} > 0$, so the two families bifurcate on the same side of $\mu = 0$ (supercritical if $a_3 > 0$ and subcritical if $a_3 < -a_2^2$). If $-a_2^2 < a_3 < 0$, then $\mu_0^{(1)} < 0$ and $\mu_0^{(2)} > 0$, so the two families bifurcate on the opposite side of $\mu = 0$ (Fig. 3).

To study the stability of the 4-cycles for the map (4.19), we consider the map

$$z' = F^4_\mu(z) = \lambda(\mu)^4 z + 4c_1 z^2 \bar{z} + 4c_2 z^3 + \mathcal{O}(|\mu||z|^3 + |z|^5).$$

(4.28)

If $\sigma_1$ and $\sigma_2$ are the eigenvalues of the Jacobian $A = \partial(z', \bar{z}')/\partial(z, \bar{z})$ at one of the 4 fixed points $x$ of one family for $F^4_\mu(z)$ and also if we
assume that we used an area-preserving transformation of the form (2.11) then we can easily see that i) \( \sigma_1 \) and \( \sigma_2 \) are complex conjugate on the unit circle for \( \mu^{(2)} \) if \( a_3 > a_2^2 \) or \( a_3 < -a_2^2 \), and also for \( \mu^{(1)} \) if \( 0 < a_3 < a_2^2 \); ii) \( \sigma_1 \) and \( \sigma_2 \) are real reciprocal each other for \( \mu^{(1)} \) if \( a_3 > a_2^2 \) or \( a_3 < -a_2^2 \), also for \( \mu^{(2)} \) if \( 0 < a_3 < a_2^2 \), and for both \( \mu^{(1)} \) and \( \mu^{(2)} \) if \( -a_2^2 < a_3 < 0 \).

From the above results, we can state the following theorem.

**Theorem 3.** Let \( F_\mu : \mathbb{C} \to \mathbb{C} \) be the map given in (2.7) and assume that \( \lambda_0^4 = 1 (\lambda_0 \neq \pm 1) \) and \( a_3 \neq 0 \). Then, generically we have two one-parameter families of 4-periodic fixed points \( \{ x^{(j)}_1(r), x^{(j)}_2(r), x^{(j)}_3(r), x^{(j)}_4(r) \} | r \in \mathbb{R}^+ \), \( j = 1, 2 \) bifurcating from the origin and those 4-periodic points are given by

\[
x_k^{(j)} = x_k^{(j)}(r) = r e^{2\pi i [\phi_0^{(j)} + \frac{k-1}{4}]} + O(r^3) \quad (j = 1, 2, k = 1, 2, 3, 4).
\]

where the parameter \( r \) is related to \( \mu \) as in (4.27).

Moreover, if \( a_3 > 0 \) or \( a_3 < -a_2^2 \), then the two families bifurcate on the same side of \( \mu = 0 \) and one family with smaller \( r \) is hyperbolic (saddle) and the other with larger \( r \) is elliptic. If \( -a_2^2 < a_3 < 0 \), then the two families bifurcate on the opposite side of \( \mu = 0 \) and both are hyperbolic (saddle).

(iv) The case \( n \geq 5 \)

when \( \lambda_0 = e^{2\pi i / n} (n \geq 5) \), the normal form of \( F_\mu(z) \) is

\[
(4.29) \quad F_\mu(z) = \lambda(\mu)z + \alpha(\mu)z^2 \bar{z} + \beta(\mu)z^{n-1} + \gamma(\mu)z^3 \bar{z}^2 + O(|z|^7 + |z|^n)
\]

and the coefficient \( \alpha(0) \equiv \alpha_0 \) can be computed from (2.12) as

\[
\alpha_0 = -\frac{\lambda_0}{8(\text{Im} \lambda_0)^2} \left[ 3i a_3 \text{Im} \lambda_0 - a_2^2 \cdot \frac{(\lambda_0 + 1)(2\lambda_0^2 + \lambda_0 + 2)}{\lambda_0^3 - 1} \right].
\]

Since

\[
(\lambda_0 + 1)(2\lambda_0^2 + \lambda_0 + 2) = \frac{\lambda_0 + 1}{\lambda_0 - 1} \left( 1 + \frac{\lambda_0^2 + 1}{\lambda_0^3 + \lambda_0 + 1} \right)
\]

\[
= \frac{\lambda_0 + 1}{\lambda_0 - 1} \left( 1 + \frac{1}{1 + \frac{1}{\lambda_0 + \lambda_0}} \right)
\]

\[
= -i \cot \frac{\pi}{n} \cdot \frac{1 + 4 \cos \frac{2\pi}{n}}{1 + 2 \cos \frac{2\pi}{n}} \quad (n \geq 5),
\]

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\( \alpha_0 \) can be rewritten as

\[
(4.30) \quad \alpha_0 = \lambda_0 i \xi_n \quad (n \geq 5),
\]

where

\[
(4.31) \quad \xi_n = -\frac{1}{16} \csc \frac{2\pi}{n} \left( 6a_3 + a_2 a_2 \cdot \csc^2 \frac{\pi}{n} \cdot \frac{1 + 4 \cos \frac{2\pi}{n}}{1 + 2 \cos \frac{2\pi}{n}} \right) \quad (n \geq 5).
\]

Notice that (4.30) also covers the case \( n = 4 \).

The bifurcation equation (3.15) becomes

\[
(4.32) \quad 2\pi i \mu z + c_1 z^2 \overline{z} + c_2 z^{n-1} + O(|\mu|^2 |z| + |\mu| |z|^3 + |\mu| |z|^{n-1} + |z|^5) = 0,
\]

where \( c_1 = \bar{\lambda}_0 \alpha_0 = i \xi_n, \quad c_2 = \bar{\lambda}_0 \beta_0. \)

Setting \( z = re^{2\pi i \varphi} \) and separating the trivial solution \( r = 0 \), we have

\[
(4.33) \quad 2\pi i \mu + i \xi_n r^2 + c_2 r^{n-2} e^{-2n\pi i \varphi} + O(|\mu|^2 + |\mu|r^2 + |\mu|r^{n-2} + r^4) = 0.
\]

For \( n = 5 \), we set

\[
(4.34) \quad \begin{cases} 
\mu = \mu_0 r^2 + \mu_1 r^3 + \mu_2 r^3 \\
\varphi = \varphi_0 + \varphi_1
\end{cases}
\]

and take \( \mu_0 \) as

\[
(4.35) \quad \mu_0 = -\frac{\xi_5}{2\pi}.
\]

Then (4.33) becomes

\[
(4.36) \quad 2\pi i \mu_1 + c_2 e^{-10\pi i \varphi} + 2\pi i \mu_2 + O(r) = 0.
\]

Now assume that \( c_2 \neq 0 \). Then we can take \( \mu_1 \) and \( \varphi_0 \) such that

\[
2\pi i \mu_1 + c_2 e^{-10\pi i \varphi_0} = 0,
\]

that is,

\[
(4.37) \quad \begin{cases} 
\mu_1 = -\frac{|c_2|}{2\pi} \\
\varphi_0 = \frac{1}{10\pi} \arg\left( \frac{c_2}{2\pi i} \right) \quad (\text{mod } \frac{1}{5})
\end{cases}
\]
From (4.36), we define
\[ h(\mu_2, \varphi, r) = 2\pi i \mu_1 + c_2 e^{-10\pi i \varphi} + 2\pi i \mu_2 + \mathcal{O}(r). \]

Then,
\[ h(0, \varphi_0, 0) = 0, \quad \frac{\partial h}{\partial \mu_2}(0, \varphi_0, 0) = 2\pi i, \]
\[ \frac{\partial h}{\partial \varphi}(0, \varphi_0, 0) = -10\pi i c_2 e^{-10\pi i \varphi_0} = 10\pi |c_2| \neq 0. \]

Hence, by the implicit function theorem, we have
\[ \mu_2 = \mu_2(r) = \mathcal{O}(r), \quad \varphi_1 = \varphi_1(r) = \mathcal{O}(r). \]

Therefore we have a one-parameter family of 5-cycles bifurcating from the origin, given by
\[
\begin{align*}
\mu &= -\frac{\xi_5}{2\pi} r^2 - \frac{|c_2|}{2\pi} r^3 + \mathcal{O}(r^4) \\
\varphi &= \frac{1}{10\pi} \arg\left(\frac{c_2}{2\pi i}\right) + \mathcal{O}(r) \quad \text{(mod } \frac{1}{5})
\end{align*}
\]
(4.38)

For \( n \geq 6 \), we set
\[ \mu = -\frac{\xi_5}{2\pi} r^2 + \mu_1 r^4, \]
and can proceed as before by imposing more conditions on the coefficients of the higher order terms.

Thus, we have the following theorem.

**Theorem 4.** Let \( F_\mu : \mathbb{C} \to \mathbb{C} \) be the map given in (2.7) and assume that
\[ \lambda_0^n = 1 \quad (\lambda_0 \neq \pm 1) \quad (n \geq 5). \]

Then, generically, we have a one-parameter family of \( n \)-periodic fixed points bifurcating from the origin.
Fig. 1. The bifurcation diagrams and the positions of the 3-periodic fixed points for $\theta_0 = \frac{1}{3}$ and $a_2 \neq 0$

(a_2 > 0, $\mu > 0$)  (a_2 > 0, $\mu < 0$)

Fig. 2. The bifurcation diagrams and the positions of the 3-periodic fixed points for $\theta_0 = \frac{1}{3}$ and $a_2 = 0$

$a_2 > 0$, $\mu > 0$

$Z_3^{(1)}$ for $\mu = \mu^{(1)}$

$Z_3^{(2)}$ for $\mu = \mu^{(2)}$
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(a) $a_1 > \tilde{a}_2$

(b) $0 < a_2 < \tilde{a}_2$
Fig. 3. The bifurcation diagrams and the positions of the 4-periodic fixed points

- $a_3 < - a_2^2$
- $- a_2^2 < a_1 < 0$

$Z_4^{(1)}$ for $\varphi = \varphi^{(1)}$
$Z_4^{(2)}$ for $\varphi = \varphi^{(2)}$
$Z_4^{(3)}$ for $\varphi = \varphi^{(3)}$
$Z_4^{(4)}$ for $\mu > 0$
$Z_4^{(5)}$ for $\mu < 0$
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References


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