E.W. Barnes’ approach of the multiple gamma functions

Junesang Choi and J. R. Quine

In this paper we provide a new proof of multiplication formulas for the simple and double gamma functions and also give some related asymptotic expansions.

1. E.W. Barnes’ definition of multiple gamma functions

In [3] E. W. Barnes introduces the multiple Hurwitz $\zeta$-function, for $\text{Re} s > r$,

$$\zeta_r(s, a| w_1, \ldots, w_r) = \sum_{m_1, m_2, \ldots, m_r = 0}^{\infty} \frac{1}{(a+\Omega)^s}$$

where $\Omega = m_1 w_1 + m_2 w_2 + \cdots + m_r w_r$ and also represents the $r$-ple Hurwitz $\zeta$-function by the contour integral

$$\zeta_r(s, a| w_1, w_2, \ldots, w_r) = \frac{i\Gamma(1-s)}{2\pi} \int_L \frac{e^{-az}(-z)^{s-1}}{\prod_{k=1}^{r}(1-e^{-w_k z})}dz$$

where the conditions for $a$ and $w_1, \ldots, w_r$ are described in [3] and the possible contour $L$ is given by

For our purpose we restrict these when \( w_k = 1, \ k = 1, 2, \ldots, n \) and the contour \( C \) is the same as Fig. I in [4]. That is to say, \( a > 0, \ \text{Re} \ s > n \),

\[
\zeta_n(s, a) = \sum_{k_1, k_2, \ldots, k_n = 0}^{\infty} (a + k_1 + k_2 + \cdots + k_n)^{-s}.
\]

Then \( \zeta_n(s, a) \) can be continued to a meromorphic function with poles \( s = 1, 2, \ldots, n, \ a > 0 \), for by the contour integral representation

\[
\zeta_n(s, a) = \frac{i\Gamma(1 - s)}{2\pi} \int_c \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-z})^n} \, dz
\]

the integral is valid for \( a > 0 \) and all \( s \), so \( \zeta_n(s, a) \) has possible poles only at the poles of \( \Gamma(1 - s) \), i.e., \( s = 1, 2, 3, \ldots \). But by the series definition \( \zeta_n(s, a) \) is holomorphic for \( \text{Re} \ s > n \) [4]. In particular, when \( n = 1 \),

\[
\zeta_1(s, a) = \sum_{k=0}^{\infty} (a + k)^{-s} = \zeta(s, a)
\]

is the well-known Hurwitz \( \zeta \)-function, which can be continued to a meromorphic function with only simple pole at \( s = 1 \) having its residue 1, by the contour integral representation [1], [4], [9].

Now we summarize some known propositions.

**Proposition 1.1.** [7]. Let \( \zeta(s, a) = \sum_{k=0}^{\infty} (k+a)^{-s} \) be the Hurwitz
ζ-function, where \( a > 0 \) and \( \operatorname{Re} s > 1 \), then we have

\[
\Gamma(a) = \frac{e^{\zeta'(0,a)}}{R_1}, \quad \text{where } R_1 \text{ is a constant and}
\]

\[
\zeta'(s, a) = \frac{\partial}{\partial s} \zeta(s, a).
\]

**Proof.** As above, by the contour integral representation of \( \zeta(s, a) \), \( \zeta(s, a) \) is analytically continued for all \( s \neq 1, (a > 0) \).

\[
\zeta(s, a + 1) = \zeta(s, a) - a^{-s},
\]

\[
\zeta'(s, a + 1) = \zeta'(s + a) + a^{-s} \log a,
\]

\[
\zeta'(0, a + 1) = \zeta'(0, a) + \log a.
\]

Letting \( G_1(a) = e^{\zeta'(0,a)} \), we have \( G_1(a + 1) = aG_1(a), a > 0 \), and

\[
\frac{d^2}{da^2} \log G_1(a) = \frac{d^2}{da^2} \frac{d}{ds} \zeta(s, a) \bigg|_{s=0}.
\]

Any by the analytic continuation of \( \zeta(s, a) \) one sees that \( G_1(a) \) is \( C^\infty \) on \( \mathbb{R}^+ \). So by the Bohr-Mollerup Theorem

\[
G_1(a) = \Gamma(a)R_1, \quad R_1 \quad \text{constant}.
\]

Note that \( R_1 = e^{\zeta'(0)} \) since \( \zeta(s, 1) = \zeta(s) \) and so

\[
R_1 = G_1(1) = e^{\zeta'(0,1)}.
\]

Now define

\[
G_n(a) = e^{\zeta'_n(0,a)}, \quad \text{where } \zeta'_n(s, a) = \frac{\partial}{\partial s} \zeta_n(s, a).
\]

The basic properties of \( G_n(a) \) are now given by the following proposition.
PROPOSITION 1.2. [7].

(a) \( G_{n+1}(a+1) = \frac{G_{n+1}(a)}{G_n(a)} \).

(b) \( G_n(a) \) can be continued a meromorphic function on \( C \) with poles at the negative integers and a simple pole at zero.

(c) Let \( R_n = \lim_{a \to 0} aG_n(a) \), then \( G_n(1) = R_n/R_{n-1} \), where \( R_0 = 1 \).

In particular, when \( n = 2 \),

COROLLARY 1.3.

(a) \( G_2(a+1) = G_2(a)/G_1(a) \).

(b) \( G_2(a) \) can be continued to a meromorphic function on \( C \) with poles at the negative integers and a simple pole at zero.

Now we can get the relationship between multiple gamma functions and multiple Hurwitz \( \zeta \)-functions.

PROPOSITION 1.4. [7].

\[ \Gamma_n(a) = \left( \prod_{m=1}^{n} R_{n-m+1}^{(m-1) \zeta_m} \right) G_n(a). \]

\textit{Proof.} See [4] and [7].

In particular, when \( n = 2 \),

COROLLARY 1.5.

\[ \Gamma_2(a) = (R_2^{-1} R_1^a) G_2(a) \]

where \( R_2 = e^{\zeta''(0)} e^{\zeta'_2(0,1)} = \lim_{a \to \infty} aG_2(a) \).

2. Multiplication formulas for \( \Gamma \) and \( \Gamma_2 \)

In this section we provide other proofs of multiplication formulas for the simple and double gamma functions.
Theorem 2.1.

\[ \prod_{l=0}^{m-1} \Gamma(ka + \frac{kl}{m}) = (2\pi)^{1/2m-1/2k}(\frac{k}{m})^{m(k+1/2(mk-m-k)}} \prod_{n=0}^{k-1} \Gamma(ma + \frac{mn}{k}), \]

\[ k, m = 1, 2, 3, \ldots. \]

Proof. Note that \( \{i = 0, 1, 2, \ldots\} = \{kj + n, 0 \leq n \leq k-1, j = 0, 1, 2, \ldots\}. \)

\[ \sum_{l=0}^{m-1} \zeta(s, ka + \frac{kl}{m}) = \sum_{l=0}^{m-1} \sum_{i=0}^{\infty} (ka + \frac{kl}{m} + i)^{-s} \]

\[ = (\frac{m}{k})^{s} \sum_{i=0}^{\infty} \sum_{l=0}^{m-1} \frac{1}{(ma + l + \frac{mn}{k})^{s}} \]

\[ = (\frac{m}{k})^{s} \sum_{l=0}^{m-1} \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \frac{1}{(ma + l + \frac{mn}{k}(kj + n))^{s}} \]

\[ = (\frac{m}{k})^{s} \sum_{n=0}^{k-1} \sum_{j=0}^{\infty} \sum_{l=0}^{m-1} \frac{1}{(ma + \frac{mn}{k} + mj + l)^{s}} \]

\[ = (\frac{m}{k})^{s} \sum_{n=0}^{k-1} \sum_{j=0}^{\infty} \frac{1}{(ma + \frac{mn}{k} + j)^{s}} \]

\[ = (\frac{m}{k})^{s} \sum_{n=0}^{k-1} \zeta(s, ma + \frac{mn}{k}). \]

\[ \sum_{l=0}^{m-1} \zeta(s, ka + \frac{kl}{m}) = (\frac{m}{k})^{s} \sum_{n=0}^{k-1} \zeta(s, ma + \frac{mn}{k}). \]

Now we have

\[ \sum_{l=0}^{m-1} \zeta'(s, ka + \frac{kl}{m}) \]

\[ = (\frac{m}{k})^{s} \log(\frac{m}{k}) \sum_{n=0}^{k-1} \zeta(s, ma + \frac{mn}{k}) + (\frac{m}{k})^{s} \sum_{n=0}^{k-1} \zeta'(s, ma + \frac{mn}{k}), \]
where the accent \( \cdot \) denotes the differentiation with respect to \( s \). We have

\[
\sum_{j=0}^{m-1} \zeta'(0, ka + \frac{kl}{m}) = \log\left(\frac{m}{k}\right) \left( \sum_{n=0}^{k-1} \zeta(0, ma + \frac{mn}{k}) \right) + \sum_{n=0}^{k-1} \zeta'(0, ma + \frac{mn}{k}).
\]

Since \( \zeta(0, a) = \frac{1}{2} - a \), we have

\[
\sum_{n=0}^{k-1} \zeta(0, ma + \frac{mn}{k}) = \sum_{n=0}^{k-1} (1/2 - ma - \frac{mn}{k}) = (1/2 - ma)k - \frac{m}{2}(k - 1).
\]

Thus

\[
\prod_{l=0}^{m-1} e^{\zeta'(0, ka + \frac{kl}{m})} = \left( \frac{m}{k} \right)^{(1/2 - ma)k - \frac{m}{2}(k - 1)} \prod_{n=0}^{k-1} e^{\zeta'(0, ma + \frac{mn}{k})}.
\]

By Proposition 1.1, \( e^{\zeta'(0, a)} = e^{\zeta(0)} \Gamma(a) \), so we have

\[
e^{m\zeta'(0)} \prod_{l=0}^{m-1} \Gamma(ma + \frac{kl}{m}) = \left( \frac{m}{k} \right)^{(1/2 - ma)k - \frac{m}{2}(k - 1)} e^{k\zeta'(0)} \prod_{n=0}^{k-1} \Gamma(ma + \frac{mn}{k}).
\]

\[
\prod_{l=0}^{m-1} \Gamma(ka + \frac{kl}{m}) = e^{(k-m)\zeta'(0)} \left( \frac{m}{k} \right)^{(1/2 - ma)k - \frac{m}{2}(k - 1)} \prod_{n=0}^{k-1} \Gamma(ma + \frac{mn}{k}).
\]

Note that

\[
\zeta'(0) = -1/2 \log(2\pi),
\]

\[
e^{(k-m)\zeta'(0)} = (2\pi)^{1/2(m-k)},
\]

\[
\left( \frac{m}{k} \right)^{(1/2 - ma)k - \frac{m}{2}(k - 1)} = \left( \frac{k}{m} \right)^{mak+1/2(mk-m-k)}.
\]

This completes the proof of Theorem 2.1.

Now we get the classical Gauss' multiplication formula as the special case of Theorem 2.1.
**Corollary 2.2.**

\[ \prod_{i=0}^{m-1} \Gamma(a + \frac{1}{m}) = (2\pi)^{1/2m-1/2}m^{1/2-\frac{ma}{m}}\Gamma(ma), \quad m = 2, 3, 4, \ldots. \]

**Proof.** Plug \( k = 1, m = 2, 3, 4, \ldots \) in the formula of Theorem 2.1.

Finally we provide another proof of the multiplication formula for \( \Gamma_2 \).

**Theorem 2.3.** [2],[4].

\[ \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) = K(2\pi)^{-n(n-1)\frac{\xi}{2}}n^{\frac{n^2-2}{2}-nx}\Gamma_2(nx) \]

where

\[ K = A^{n^2-1}c^{\frac{1-n^2}{2\pi}}(2\pi)^{\frac{(n-1)\xi}{2}}n^{\frac{\xi}{2}}. \]

**Lemma.**

\( \zeta_2(s, x) = \zeta(s - 1, x) + (1 - x)\zeta(s, x). \)

In particular, \( \zeta_2(s, 1) = \zeta(s - 1) \) and so \( \zeta_2'(0, 1) = \zeta'(-1). \)

Note that the classical result is (See Chapter 1 in [4])

\[ \zeta(-m, x) = -\frac{B_{m+1}(x)}{m+1}, \quad m = 0, 1, 2, \ldots \]

\[ B_0(x) = 1, \]

\[ B_1(x) = x - \frac{1}{2}, \]

\[ B_2(x) = x^2 - x + \frac{1}{6}, \]

\[ : \]

**Proof of Lemma.** Note that

\[ \zeta_n(s, x) = \sum_{k_1, \ldots, k_n=0}^{\infty} (x + k_1 + k_2 + \cdots + k_n)^{-s} = \sum_{k=0}^{\infty} \frac{(k+n-1)}{(x+k)^s}. \]
This is because the number of solutions of $k_1 + k_2 + \cdots + k_n = k$, $k = 0, 1, 2, \ldots, (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ is equal to the coefficient of $x^k$ in the expansion of the Maclaurin series of $(1 - x)^{-n}$, i.e., $\binom{k + n - 1}{n - 1}$.

In particular,

$$
\zeta_2(s, x) = \sum_{k_1, k_2 = 0}^{\infty} (x + k_1 + k_2)^{-s}
= \sum_{k=0}^{\infty} \frac{k + 1}{(x + k)^{s}}
= \sum_{k=0}^{\infty} \frac{(x + k)}{(x + k)^{s}} + \sum_{k=0}^{\infty} \frac{1 - x}{(x + k)^{s}}
= \zeta(s - 1, x) + (1 - x)\zeta(s, x).
$$

Thus we have $\zeta_2(s, x) = \zeta(s - 1, x) + (1 - x)\zeta(s, x)$.

**Proof of Theorem 2.3.** Consider

$$
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \zeta_2(s, x + \frac{i+j}{n})
= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k_1, k_2 = 0}^{\infty} (x + i + j + k_1 + k_2)^{-s}
= n^s \sum_{k_1, k_2 = 0}^{\infty} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (nx + i + j + nk_1 + nk_2)^{-s}
= n^s \sum_{k_1, k_2 = 0}^{\infty} (nx + k_1 + k_2)^{-s}
= n^s \zeta_2(s, nx).
$$

$$
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \zeta_2(s, x + \frac{i+j}{n}) = (\log n)n^s \zeta_2(s, nx) + n^s \zeta_2'(s, nx),
$$
where the accent denotes the differentiation with respect to \( s \). Therefore we have

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \zeta'_2(0, x + \frac{i+j}{n}) = (\log n) \zeta_2(0, nx) + \zeta'_2(0, nx).
\]

\[
\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \frac{\zeta'_2(0, x + \frac{i+j}{n})}{\zeta'_2(0, nx)} = n^{\zeta_2(0, nx)} \zeta'_2(0, nx).
\]

\[
\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} G_2(x + \frac{i+j}{n}) = n^{\zeta_2(0, nx)} G_2(nx).
\]

Note that \( G_2(x) = R_2 R_1^{-a} \Gamma_2(x) \), by Corollary 1.5.

\[
R_2^{n^2} \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) R_1^{-\Gamma_2(x + \frac{i+j}{n})} = n^{\zeta_2(0, nx)} R_2 R_1^{-nx} \Gamma_2(nx).
\]

\[
\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} R_1^{-(x + \frac{i+j}{n})} = R_1^{-\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (x + \frac{i+j}{n})}
\]

\[
= R_1^{-(n^2 x + n^2 - n)}.
\]

Thus we have

\[
\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) = F(x, n) \Gamma_2(nx)
\]

where \( F(x, n) = n^{\zeta_2(0, nx)} R_2^{1-n^2} R_1^{n^2 x - nx + n^2 - n} \).

\[
R_1 = e^{\zeta'(0)} = e^{-1/2 \log(2\pi)} = (2\pi)^{-1/2}.
\]

\[
R_2 = e^{\zeta'(0)} e^{\zeta'_2(0, 1)} = (2\pi)^{-1/2} e^{\zeta'_2(-1)} \text{ (Lemma)} = (2\pi)^{-1/2} e^{1/2} \text{ (see Chapter 2 in [4])}.
\]
\[ \zeta_2(0, nx) = \zeta(-1, nx) + (1 - nx)\zeta(0, nx) \]
\[ = -1/2n^2x^2 + 1/2nx - \frac{1}{12} + (1 - nx)(1/2 - nx) \]
\[ = 1/2n^2x^2 - nx + \frac{5}{12}. \]

Then we have
\[ F(x, n) = n^{1/2n^2x^2 - nx + \frac{\delta}{2}}(2\pi)^{-1/2(1-n^2)}A^{n^2 - 1}e^{\frac{1}{12}(1-n^2)} \]
\[ \times (2\pi)^{-1/2(n^2x-nx)-1/2(n^2-n)} \]
\[ = K(2\pi)^{-n(n-1)\frac{\delta}{2} n^{2x^2 - nx}}. \]

where \( K = A^{n^2}e^{\frac{1-n^2}{12}}(2\pi)^{n-1} n^{\delta n}. \)

Therefore we have
\[ \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) = K(2\pi)^{-n(n-1)\frac{\delta}{2} n^{x^2 - nx}} \]
We make \( x = 0 \) in the formula just obtained.

**Corollary 2.4.**
\[ \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2\left(\frac{i+j}{n}\right) = \frac{K}{n} \]

where the accent' denotes that we remove the case \( i = 0, j = 0. \)

**Proof.** Note that
\[ \lim_{x \to 0} \frac{\Gamma(nx)}{\Gamma(x)} = \frac{1}{n}. \]

From the first expression on \( \Gamma_2^{-1}(x + 1), \)
\[ \frac{\Gamma(x)}{\Gamma_2(x)} = \frac{1}{\Gamma_2(x + 1)} = A(x) \]
where
\[ A(x) = (2\pi)^{\frac{\delta}{2}}e^{-\frac{x(x+1)}{2}} - \frac{x^2}{2} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right)^k e^{-x + \frac{\delta^2}{2x}}. \]
Then we have

$$\Gamma(x) = A(x) \Gamma_2(x), \quad \lim_{x \to 0} A(x) = 1 = \lim_{x \to 0} A(nx).$$

$$\lim_{x \to 0} \frac{\Gamma_2(nx)}{\Gamma_2(x)} = \lim_{x \to 0} \frac{\Gamma(nx)}{A(nx) \Gamma(x)} = \lim_{x \to 0} \frac{\Gamma(nx)}{\Gamma(x)}.$$

$$\lim_{x \to 0} \frac{xe^{\gamma x}}{nx \gamma} \prod_{k=1}^{\infty} \left(1 + \frac{k}{x}\right) e^{-\frac{k}{x}} = \frac{1}{n}. \quad \lim_{x \to 0} \frac{f_2(nx)}{f_2(x)} = \lim_{x \to 0} f_2(nx) A(nx) = \lim_{x \to 0} f_2(nx).$$

$$\prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) = \Gamma_2(x) \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}).$$

$$\lim_{x \to 0} \prod_{i=0}^{n-1} \prod_{j=0}^{n-1} \Gamma_2(x + \frac{i+j}{n}) = K \lim_{x \to 0} \frac{\Gamma_2(nx)}{\Gamma_2(x)} = \frac{K}{n}.$$

3. Some related asymptotic expansions

**Proposition 3.1.**

(a) \(\frac{\partial}{\partial s} \zeta(s, a)|_{s=0} = \log \Gamma(a) - 1/2 \log(2\pi)\)

(b) For real \(a > 0\),

$$\frac{\partial}{\partial s} \zeta(s, a)|_{s=0} = (a - 1/2) \log a - a + \frac{\theta(a)}{12a}.$$

where \(0 < \theta(a) < 1\).

**Proof.**

(a) We know that \(\Gamma(a) = e^{\zeta'(0,a)} / R_1\), from Proposition 1.1, where \(R_1 = e^{\zeta'(0)}\). Thus

$$\zeta'(0,a) = e^{\zeta'(0)} \Gamma(a) = (2\pi)^{-1/2} \Gamma(a) [4],[5].$$

$$\zeta'(0,a) = \frac{\partial}{\partial s} \zeta(s, a)|_{s=0} = \log \Gamma(a) - 1/2 \log(2\pi).$$

(b) It is known [1] that for real \(x > 0\),

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x} e^{\frac{\theta(x)}{12x}} \text{ with } 0 < \theta(x) < 1.$$
Then \( \log \Gamma(a) = \frac{1}{2} \log(2\pi) + (a - \frac{1}{2}) \log a - a + \frac{\theta(a)}{12a} \).

Therefore, as in (a),

\[
\zeta'(0, a) = (a - 1/2) \log a - a + \frac{\theta(a)}{12a},
\]

for real \( a > 0 \), where \( 0 < \theta(a) < 1 \).

**PROPOSITION 3.2.**

(a) \( \frac{\partial}{\partial s} \zeta_2(s, x)|_{s=0} = \log \Gamma_2(x) + 1/2(x - 1) \log(2\pi) + \frac{1}{12} - \log A \)

where \( A \) is the Kinkelin's constant.

(b) For real \( x > 0 \),

\[
\frac{\partial}{\partial s} \zeta_2(s, x)|_{s=0} = \frac{3x^2}{4} - x - \left(\frac{x^2}{2} - x + \frac{5}{12}\right) \log x + O\left(\frac{1}{x}\right), \ x \to \infty.
\]

**Proof.** (a) We know that

\[
\Gamma_2(x) = (R_2^{-1} R_1^x) e^{\zeta'(0, x)},
\]

where \( R_1 = e^{\zeta'(0)} \), \( R_2 = e^{\zeta'(0)} e^{\zeta_2(0, 1)} \), Corollary 1.5.

Then we have

\[
\zeta_2'(0, x) = \frac{\partial}{\partial s} \zeta_2(s, x)|_{s=0}
= \log \Gamma_2(x) + (\log R_2 - x \log R_1)
= \log \Gamma_2(x) + (1 - x) \zeta'(0) + \zeta_2'(0, 1)
= \log \Gamma_2(x) + 1/2(x - 1) \log(2\pi) + \zeta'(-1),
\]

since \( \zeta_2'(0, 1) = \zeta'(-1) \), \( \zeta'(0) = -1/2 \log(2\pi) \).

\[
\zeta_2'(0, x) = \log \Gamma_2(x) + 1/2(x - 1) \log(2\pi) + 1/12 - \log A
\]

since \( A = e^{1/12} - \zeta'(-1) \) and so \( \zeta'(-1) = 1/12 - \log A \).
E. W. Barnes' approach of the multiple gamma functions

(b) From Stirling's formula, for real $x > 0$,

$$\log \Gamma_2(x + 1)^{-1} = \frac{x}{2} \log 2\pi - \log A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12}\right) \log x + O\left(\frac{1}{x}\right), \ x \to \infty.$$ 

We know that $\Gamma_2^{-1}(x + 1) = \Gamma_2^{-1}(x) \Gamma(x)$.

Thus $\log \Gamma_2^{-1}(x) = \log \Gamma_2^{-1}(x + 1) - \log \Gamma(x)$.

In the course of proof of (b), Proposition 3.1.

$$\log \Gamma(x) = \frac{1}{2} \log(2\pi) + (x - 1/2) \log x - x + O\left(\frac{1}{x}\right), \ x \to \infty, \ x > 0.$$ 

Therefore we have

$$\log \Gamma_2^{-1}(x) = \left(\frac{x}{2} - 1/2\right) \log(2\pi) + \left(\frac{x^2}{2} - x + \frac{5}{12}\right) \log x
- \frac{3}{4} x^2 + x - \log A + \frac{1}{12} + O\left(\frac{1}{x}\right), \ x \to \infty.$$ 

From (a),

$$\partial / \partial s \zeta_2(s, x)|_{s=0} = \frac{3}{4} x^2 - x - \left(\frac{x^2}{2} - x + \frac{5}{12}\right) \log x + O\left(\frac{1}{x}\right), \ x \to \infty.$$ 

References

3. ________, On the theory of the multiple gamma function, Philosophical Transactions of the Royal Society (A), XIX (1904), 374–439.

Department of Mathematics  
Dongguk University  
Kyoung Ju 780–350, Korea

Department of Mathematics  
Florida State University  
Tallahassee, Florida 32306  
U.S.A.