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Bootstrap Confidence Cones for Spherical Data

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ABSTRACT

The set of eigenvectors of the second moment matrix and the mean vector are the measures of orientation for a distribution supported on the unit sphere. Bootstrap confidence cone for the eigenvector is constructed and the consistency of this method is discussed. The performance of our bootstrap cone for the eigenvector is compared with that of the asymptotic confidence cones for two measures under the parametric assumptions for the underlying distributions and that of the bootstrap cone for the mean vector by Monte Carlo simulation.

1. Introduction

For a distribution supported on the unit sphere, the mean vector and the set of eigenvectors of the second moment matrix are the conventional measures of orientation. As the measure of orientation, the mean vector has drawn most attention, but the mean vector representation is not adequate for spherical distributions with vanishing mean vector. The eigenvector representation of orientation is adequate even when the mean vector vanishes. Furthermore, when there is enough symmetry, the mean vector and the eigenvector coincide. Kim [5] and Watson [8] discussed the usefulness of this measure.

Let X be a random vector on the unit sphere S^2 in R^3 with distribution F. An eigenvector of the second moment matrix $M_F = EXX'$ of X is denoted by $e(F)$. A consistent estimator of $e(F)$ is $e(\widehat{F_n})$, where $\widehat{F_n}$ is empirical distribution based on observed values of the random sample $x_1, \ldots, x_n$ from F. A confidence region of

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level $(1-\alpha)$ for $e(F)$ is the cone, C, with axis $e(F)$ and semi-angle $\phi_\alpha(F)$, defined by

$$C = \{v \in S^2 \mid |v'e(\widehat{F_n})| \geq \phi_\alpha(F)\}$$

where $\phi_\alpha(F)$ is chosen so that

$$P\{e(\widehat{F_n}) \in C\} = 1 - \alpha.$$

Solutions for $\phi_\alpha(F)$ have been proposed, but they require imposing certain parametric assumptions on F; with these assumptions it is shown that the pivotal quantity

$$P(\widehat{F_n}, F) = 2n[1 - (e(\widehat{F})'e(\widehat{F_n}))^2]$$

has a nondegenerate asymptotic distribution and it is possible to estimate $\phi_\alpha(F)$ by $\phi_\alpha(\widehat{F_n})$.

In this paper a non-parametric method of constructing confidence cone for $e(F)$ via bootstrapping is proposed. In section 2, classical confidence cones for two measures and Bootstrap confidence cone for the mean vector (Ducharme Met al. [3]) are reviewed. In section 3, the bootstrap procedure for approximating the sampling distribution of $P(\widehat{F_n}, F)$ is presented and shown to be consistent under minimal conditions. In section 4, the coverage probability of our bootstrap confidence cone is compared with the coverage probability of the classical confidence cones and that of Ducharme's bootstrap confidence cone for the mean vector by Monte Carlo simulations.

2. Confidence Cones

In this section, we discuss classical confidence cones for measures of orientation for spherical distributions and review the Bootstrap confidence cone for the mean vector (Ducharme Met al. [3]).

2.1 Confidence Cone for Mean Vector

For spherical random vector X with distribution F, the mean direction vector μ_F is defined by $\mu_F = \dfrac{E_F(X)}{\| E_F(X) \|}$ where $\| E_F(X) \|$ is called the mean resultant length.

Let X be a random vector on $S^2 = \{x; \| x \| = 1, x \in R^3\}$ with density function of the form $f(x) = g(\mu' x)$ where $\| \mu \| = 1$ and assume that $EX \neq 0$. Let $x_1, \ldots, x_n$ be a random sample from X and put $S = x_1 + \ldots + x_n$. We can write $S = (\mu\mu')S + (I - \mu\mu')S$; $(I - \mu\mu')S$ is the orthogonal projection of S on the plane perpendicular to μ. To describe the accuracy of $\hat{\mu} = \dfrac{S}{\| S \|}$ as an estimator for μ, we need to consider the pivotal quantity $P(\widehat{F_n}, F) = n(1 - \mu'\hat{\mu})$.

Theorem 2.1. $P(\widehat{F_n}, F) \xrightarrow{d} \dfrac{1 - Et^2}{4(Et)^2} \chi_2^2$.

Since we do not know Et and Et^2 in practice, we may replace them by their consistent estimators; for example, we can use $\widehat{Et} = \dfrac{\| S \|}{n}$ and $\widehat{Et^2} = \dfrac{\sum_{i=1}^n (\hat{\mu}' x_i)^2}{n}$.

Hence, under the rotational symmetry, an approximate level $(1 - \alpha)$ confidence cone for the mean vector μ is given by

$$C_{RS} = \{v \in S^2 \mid v'\hat{\mu} \geq 1 - \frac{n[1 - \frac{1}{n}\sum_{i=1}^{n}(\mu' x_i)^2]}{4 \| S \|^2} c_\alpha\}$$

where c_α is the upper α-percentage point of χ_2^2.

2.2 Confidence Cone for Eigenvector

Viewing a probability distribution on S^2 as a mass distribution on S^2, we may make an analogy with the problem of finding the orientation of a rigid body. It is customary to use the orthonormal set of eigenvectors of the inertia matrix of a rigid body as its measure of orientation. When n points with unit masses are placed at $x_1, \ldots, x_n$, the moment of inertia of these masses about direction $a(\| a \|) = 1)$ is given by $a(I - \sum_{i=1}^n x_i x_i')a$ and the inertia matrix $I - \sum_{i=1}^n x_i x_i'$ is the basic entity for the rigid motion of these masses. Since the spectral decomposition of $I - [\sum_{i=1}^n x_i x_i']$ and $\sum_{i=1}^n x_i x_i'$ are obviously related, given a probability distribution F on S^2 we may correspondingly consider a simpler matrix $M_F = E_F XX'$, the second moment matrix.

For the random sample $x_1, \ldots, x_n$ from F, we can define the sample version

$M_{\hat{n}}$ by

$$M_{\hat{n}} = M_{\widehat{F_n}} = E_{\widehat{F_n}} XX' = \frac{1}{n}\sum_{i=1}^{n} x_i x_i'$$

which can be served as a consistent estimator for M.

Let X be a random vector on S^2 under the rotational symmetry and allow that $EX = 0$. Since we are restricting our attention to non-uniform rotationally symmetric distributions, the spectral decomposition of M is clearly given by

$$M = \lambda_1 \mu_1 \mu_1' + \lambda_2 P_{\perp}.$$

where λ_1 is the largest eigenvalue of multiplicity one and μ_1 is the normalized eigenvector corresponding to λ_1 which happens to lie in the direction of the axis of symmetry; λ_2 is the smaller eigenvalue of multiplicity 2 and $P_{\perp}$ is the orthogonal projection into the plane perpendicular to the axis of symmetry.

Applying the pertubation theory with random pertubation (Kim [5], Watson [8]),

$$M = \sum_{j=1}^{r} \lambda_j P_j (r \leq p)$$

where $\lambda_1, \ldots, \lambda_r > 0$ are the positive distinct eigenvalues of M and the $P_1, \ldots, P_r$ are the corresponding eigenprojections.

Let X be a random vector on S^2 with density function on the form $f(x) = g(\mu' x)$ where $\| \mu \| = 1$ and x can be written as $x = (\mu\mu') + (I - \mu\mu')x$. Put $t = \mu' x$. Then

$$M = EXX' = (Et^2)\mu\mu' + \frac{1 - Et^2}{2}(I - \mu\mu').$$

Hence, writing $x_i = t_i \xi_i + (1 - t_i^2)^{\frac{1}{2}} \eta_i$ where $\xi_i = \mu\mu' x_i$ and $\eta_i = (I - \mu\mu')x_i$.

To describe the accuracy of $\hat{\mu}$ as an estimator for μ, we need to consider the pivotal quantity

$$P(\widehat{F_n}, F) = 2n(1 - (\mu' \hat{\mu})^2).$$

Theorem 2.2. $P(\widehat{F_n}, F) \xrightarrow{d} \dfrac{4(Et^2 - Et^4)}{(3Et^2 - 1)^2} \chi_2^2$

We can use

$$\widehat{Et^2} = \sum_{i=1}^{n} (\hat{\mu}' x_i)^2/n, \quad \widehat{Et^4} = \sum_{i=1}^{n} (\hat{\mu}' x_i)^4/n$$

as the consistent estimators of Et^2 and Et^4. Hence, under the rotational symmetry, an approximate level $(1-\alpha)$ confidence cone for the eigenvector μ is given by

$$C_{AS} = \{v \in S^2 \mid v'\hat{\mu} \geq 1 - \frac{4(\sum(\hat{\mu}'x_i)^2 - \sum(\hat{\mu}'x_i)^4)}{2n(\sum(\hat{\mu}'x_i^2 - n)^2)} c_\alpha\}$$

where c_α is the upper α-percentage point of χ_2^2.

2.3 Bootstrap Procedure

Let $x_1, \ldots, x_n$ be iid random vectors with unknown c.d.f. F. Efron [4] discusses a "Bootstrap method" for approximating the sampling distribution of a function of the observations and the underlying distribution F alias the pivot. We call the approximation as the bootstrap distribution.

Let $P(\widehat{F_n}, F)$ be a real valued function of $x_1, \ldots, x_n$ and F, where $\widehat{F_n}$ is the empirical c.d.f. putting mass $\frac{1}{n}$ at $x_i, i = 1, \ldots, n$. Let $J_n(\cdot, F)$ denote the c.d.f. of $P(\widehat{F_n}, F)$. Then the bootstrap estimator of $J_n(\cdot, F)$ is $J_n(\cdot, \widehat{F_n})$ which is the c.d.f. of $P(\widehat{Fn}^*, F)$ where $\widehat{F_n}^*$ is the empirical c.d.f. of the random sample drawn from $\widehat{F_n}$. Since $\widehat{F_n}$ is a discrete distribution, for large values of n, Monte Carlo approximation to $J_n(\cdot, \widehat{F_n})$ is used.

Beran [1] gives general conditions under which the bootstrap distribution as an estimator for the sampling distribution of the pivot is consistent.

Theorem 2.3. [Beran 1984] Let $\mathcal{C}_F$ be a set of sequences $\{F_n \in \mathcal{F} | n \geq 1\}$ such that $P_r[\{\widehat{F_n}\} \in \mathcal{C}_F] = 1$. Suppose that for every sequence $\{F_n\} \in \mathcal{C}_F, J_n(\cdot, F_n) \to J(\cdot, F)$ a limit distribution depending only on $\mathcal{C}_F$. Then $\{J_n(\cdot, \widehat{F_n})\}$ converges weakly to $J(\cdot, F)$ w.p. 1.

Also, the following lemma, due to Beran [1], gives general conditions under the level of a bootstrap confidence region close to the desired level, when the sample size is large.

Lemma 2.4. [Beran 1984] If $J(\cdot, F)$ is continuous in x, then

$$\lim_{n\to\infty} P_r[P(\widehat{F_n}, F) \leq c_n(\alpha, \widehat{F_n})] = 1 - \alpha$$

for every $\alpha \in (0,1)$.

2.4 Bootstrap Confidence Cone for Mean Vector

Ducharme Met al. [3] applies the results of Beran [1] to construct Bootstrap confidence cone for the mean vector of a spherical distributions.

Let X be a random vector on the unit p - dimensional sphere S^p with mean directional vector $\theta(F) = \dfrac{E_F(X)}{\| E_F(X) \|}$ where $E_F(X) = \int x dF(x)$ and $\| \cdot \|$ is the Euclidean norm.

Consider the pivotal quantity

$$P(\widehat{F_n}, F) = n(1 - \langle \theta(F), \theta(\widehat{F_n}) \rangle)$$

where $\langle \cdot \rangle$ is the usual scalar product on R^p, and $\theta(\widehat{F_n}) = \dfrac{\overline{x_n}}{\| \overline{x_n} \|}$ is the sample mean direction vector. Suppose that $E(X) \neq 0$ and consider the function $g_F : R^p \longrightarrow R$, defined by $g_F(X) = \langle E(X), X \rangle / (\| E(X) \| \cdot \| X \|)$. When $X \neq 0$, $g_F(x)$ is a continuous function satisfying $g_F(E(X)) = 1$. Let $H(F)$ denote the Hessian matrix of g_F evaluated at $E(X)$ and $B(F)$ denotes the matrix whose columns are a set of normalized eigenvectors of $H(F)$. Let $V(\cdot)$ be the variance - covariance operator. The $p \times p$ diagonal matrix $\Delta(F) = Diag[0, -\| E(X) \|^{-2}, \cdots, -\| E(X) \|^{-2}]$ satisfies $H(F) = B(F)\Delta(F)B(F)$.

Let $J_n(\cdot, \widehat{F_n})$ be the bootstrap distribution of $P(\widehat{F_n}, F)$ under $\widehat{F_n}$ and $J(\cdot, F)$ be the distribution of the random variable $\dfrac{1}{2 \| E(x) \|^2} \sum_{i=2}^{p} z_i^2$ where $(z_1, \cdots, z_p) \sim N(o, B^t(F)V(F)B(F))$. Then we have the following theorem.

Theorem 2.5. [Ducharme Met al. 1985] $J(\cdot, \widehat{F_n}) \bullet w \longrightarrow J(\cdot, F)$ $w.p.1$,

A confidence region for $\theta(F)$ is the cone C_B, with axis $\theta(\widehat{F_n})$ and semi-angle $\phi_\alpha(F)$ defined by $C = \{v \in S^p \mid \langle v, \theta(\widehat{F_n}) \rangle \geq \phi_\alpha(F)\}$ where $\phi_\alpha(F)$ is chosen so that $P_r[\theta(F) \in C] = 1 - \alpha$. If C_{B1} is the Bootstrap confidence region for $\theta(F)$ $C_{B1} = \{v \in S^p \mid \langle v, \theta(\widehat{F_n})\rangle \geq 1 - \frac{1}{n}c_\alpha\}$ where c_α is an upper α-percentage point of $J_n(\cdot, \widehat{F_n})$.

The limiting distribution $J(\cdot, F)$ is continuous in x and strictly increasing whenever $rank(V(F)) \geq 1$ because $g_F(\cdot)$ is continuous function. By lemma 2.4 (Beran), $\lim_{n\to\infty} P_r[\theta(F) \in C_{B1}] = 1 - \alpha$ for all $F \in \mathcal{F}$ such that $E(X) \neq 0$.

3. Bootstrap Confidence Cone for Eigenvectors

Bootstrap confidence regions for eigenvectors of a covariance matrix has been treated by Beran and Srivastava [2].

In this section we proceed to construct bootstrap confidence cone for the eigenvector to a spherical distribution and we will be able to exhibit the limiting distribution of the bootstrap distribution.

Let $\mathcal{F}$ be the class of all c.d.f. s F on S^2 and let X be a random vector on S^2 with c.d.f. F. We will assume through this section that F is rotationally symmetric and hence the second moment matrix M has the decomposition $M = EXX' = (Et^2)\mu\mu' + \dfrac{1-Et^2}{2}(I - \mu\mu')$ where $t = \mu' x$. For any $p \times p$ symmetric matrix $A = \{a_{ij}\}$, let $uvec(A)$ denote the $\dfrac{p(p+1)}{2}$ dimensional column vector $\{\{a_{ij}; 1 \leq i \leq j\} 1 \leq j \leq p\}$ formed from the elements in the upper triangular half of A including diagonal elements. Suppose that $Z_F = \{z_{F,ij}\}$ is a symmetric $p \times p$ random matrix such that the distribution of $uvec(Z_F)$ is normal with mean vector zero and covariance matrix Ω_F; the components of Ω_F are determined by the requirement, $cov(z_{F,ij}, z_{F,kl}) = cov_F(x_{1i} \times x_{1j}, x_{1k} \times x_{1l}), 1 \leq i,j,k,l \leq p$. If F has finite fourth moments, the distributions of $\{\sqrt{n}(M_n - M_F) \mid F : n \geq 1\}$ converge weakly to $N(0, \Omega_F)$.

Consider the pivotal quantity

$$P(\widehat{F_n}, F) = n(1 - [e'(\widehat{F_n})e(F)]^2)$$

where $e(F)$ is the non-degenerate eigenvector of M and $e(\widehat{F_n})$, the corresponding eigenvector of $M_{\widehat{n}}$.

Lemma 3.1. $\| P(M_{\hat{n}}) - P(M_F) \| = 2[1 - e'(\widehat{F_n})e(F)]$ where $P(M_F) = e(F)e(F)'$ and $P(M_{\hat{n}}) = e(\widehat{F_n})e(\widehat{F_n})'$.

Proof.

$$\| P(M_{\hat{n}}) - P(M_F) \|^2 = tr[P(M_{\hat{n}}) - P(M_F)][P(M_{\hat{n}}) - P(M_F)].$$

By idempotency and symmetry of the eigenprojections $\| P(M_{\hat{n}}) - P(M_F) \|^2 = 2[1 - e'(\widehat{F_n})e(F)]$.

Now we will proceed to find the sampling distribution $J_n(\cdot, F)$ of the above pivot.

Theorem 3.2. $J(\cdot, \widehat{F_n}) \bullet w \longrightarrow J(\cdot, F)$ $w.p.1$ where $J(\cdot, F)$ is the distribution of the random variable $\frac{1}{2} \| \dot{P}(M_F) uvec(Z_F) \|$.

Proof. Define $\mathcal{C}_F$ as the set of all sequences $\{F_n, n = 1, 2, \ldots\}$ of c.d.f. s in $\mathcal{F}$ such that

$\{F_n\} \bullet w \rightarrow F$

$\lim_{n\to\infty} E(XX'XX' \mid F_n) = E(XX'XX' \mid F)$ for the data set $\{x_i, i = 1, \ldots, n\}$

$trM_n = 1$.

By the Glivencko–Cantelli theorem, $\widehat{F_n} \rightarrow F$ $w.p.1$. Since the sphere is compact all moments of F exists and by the SLLN $\lim_{n\to\infty} E(XX'XX' \mid \widehat{F_n}) = E(XX'XX' \mid F)$. Since $\widehat{F_n}$ is a distribution on S^2, $trM_{\hat{n}} = 1$. Hence

$$P_r[\{\widehat{F_n}, n = 1, 2, \ldots\} \in \mathcal{C}_F] = 1. \qquad (1)$$

Let $\{F_n\} \in \mathcal{C}_F$ and $x_{1n}, \ldots, x_{nn}$ be a sample from F_n, with $\widehat{F_n}$. Since $M_n = (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'$,

$$\sqrt{n} M_n = \sqrt{n}[\sum_{i=1}^{n}(x_i - \bar{x_n})(x_i - \bar{x_n})']/n$$

$$= \sqrt{n}^{-1} \sum_{i=1}^{n} \{(x_i - \mu_{F_n}) - (\bar{x_n} - \mu_{F_n})\}\{(x_i - \mu_{Fn})(\bar{x_n} - \mu_{F_n})\}'$$
$$= \sqrt{n}^{-1} \sum_{i=1}^{n} (x_i - \mu_{F_n})(x_i - \mu_{F_n})' - \sqrt{n}(\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'.$$

Let $Z_{n,i} = (x_i - \mu_{F_n})(x_i - \mu_{F_n})' - (\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'$,

$$\sqrt{n}(M_{\hat{n}} - M_n) = \sqrt{n}^{-1} \sum_{i=1}^{n} Z_{n,i} - \sqrt{n}(\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})'$$

Let a be any constant column vector of dimensional $\frac{p(p+1)}{2}$. Let H_n be the c.d.f. of $a'uvec(Z_{n,i})$ under F_n and let H be the c.d.f. of $a'uvec[x_i x_i' - M]$ under F. Since $\{F_n\} \in \mathcal{C}_F$,

$$H_n \to H \text{ and } \lim_{n\to\infty} \int y^2 dH_n(y) = \int y^2 dH(y) = a'\Omega_F a.$$

Thus

$$\lim_{n\to\infty} E_{F_n}[a'uvec(Z_{n,i})]^2 = a'\Omega_F a < \infty$$

and

$$\lim_{n\to\infty} E_{F_n}[a'uvec(Z_{n,i})]^2 I[a'uvec(Z_{n,i}) > \sqrt{n}\delta] = 0$$

for every positive δ. By the Linderberg C.L.T. for a triangular array

$$a'uvec(n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_{n,i} \mid F_n) \longrightarrow a'uvec(Z_F) \sim N(0, a'\Omega_F a).$$

Hence

$$[n^{-\frac{1}{2}} \sum_{i=1}^{n} Z_{n,i} \mid F_n] \longrightarrow Z_F \tag{2}$$

where $Z_F \sim N(0, \Omega_F)$.

Since $\mathcal{L}[\sqrt{n}(\bar{x_n} - \mu_{F_n}) \mid F_n] \longrightarrow N(0, \Omega_F)$,

$$\sqrt{n}(\bar{x_n} - \mu_{F_n})(\bar{x_n} - \mu_{F_n})' \longrightarrow 0. \tag{3}$$

From (2), (3),

$$\mathcal{L}[\sqrt{n}(M_{\hat{n}} - M_n) \mid F_n] \longrightarrow N(0, \Omega_F). \tag{4}$$

Since $P(M)$ is continuously differentiable function of $uvec(M)$, by expanding $P(M_{\hat{n}})$ about M_n, we have

$$P(M_{\hat{n}}) = P(M_n) + \dot{P}(M_n)(M_{\hat{n}} - M_n) + O_P(\| M_{\hat{n}} - M_n \|).$$

Thus

$$\begin{aligned} P(\widehat{F_n}, F_n) &= n\{1 - [e'(\widehat{F_n})e(F_n)]^2\} \\ &= \frac{1}{2} \| \dot{P}(M_n\sqrt{n}(M_{\hat{n}} - M_n) + O_P(\| \sqrt{n}(M_{\hat{n}} - M_n) \|) \|^2 \\ &= \frac{1}{2} \| \dot{P}(M_n)\sqrt{n}(M_{\hat{n}} - M_n) + O_P(1) \|^2 \end{aligned}$$

Hence by Slusky lemma and (4)

$$P(\widehat{F_n}, F_n) \longrightarrow \frac{1}{2} \| P(M_F)uvec(Z_F) \|^2 \tag{5}$$

where $uvec(Z_F) \sim N(0, \Omega_F)$.

From (1), (5) and theorem 2.3, w.p.1

$$J_n(\cdot, \widehat{F_n}) \bullet w \longrightarrow \mathcal{L}[\frac{1}{2} \| \dot{P}(M_F)uvec(Z_F) \|^2]$$

where $uvec(Z_F) \sim N(0, \Omega_F)$.

A confidence region for $e(F)$ is the cone C, with axis $e(\widehat{F_n})$ and semi angle $\phi_\alpha(F)$ defined by

$$C = \{v \in S^2 \mid (v'e(\widehat{F_n}))^2 \geq \phi_\alpha(F)\}$$

where $\phi_\alpha(F)$ is chosen so that $Pr[e(F) \in C] = 1 - \alpha$. If C_{B2} is the Bootstrap confidence region for $e(F)$,

$$C_{B2} = \{v \in S^2 \mid (v'e(\widehat{F_n}))^2 \geq 1 - \frac{1}{2n}c_\alpha\}$$

where c_α is an upper α-percentage point of $J_n(\cdot, \widehat{F_n})$.

The limiting distribution $J(\cdot, F)$ is continuous in X and strictly increasing increasing because $uvec(Z_F)$ is continuous function. By lemma 2.4, $\lim_{n\to\infty} P_r[e(F) \in C_{B2}] = 1 - \alpha$ for all $F \in \mathcal{F}$.

4.Comparison

A Monte Carlo simulation was performed to study and compare the performance of our bootstrap method with the other methods in getting confidence cones of measure of orientation from spherical distribution.

The following four confidence cones based on n observations from the spherical distribution are compared in Monte Carlo study:

C_{B1}; simple bootstrap confidence cone with mean vector as the measure of orientation

C_{B2}; our simple bootstrap confidence cone with eigenvector as the measure of orientation

C_{RS}; classical confidence cone with mean vector as the measure of orientation

C_{AS}; classical confidence cone with eigenvector as the measure of orientation under the assumption that F is antipodally symmetric.

We choose to look at the case $p = 3$. Sample were drawn from different distributions with known $\mu(F)$ and $\alpha = 0.05$.

For the first simulation, we took pseudo-random samples from the von Mises-Fisher distribution with $\mu(F) = (0, 0, 1)$ and $k = 0.1$, 1 and 3; $f(x) = cexp(kx_3)$ where c is the normalizing constant.

$$f(\theta, \phi) = c(exp(kcos\theta))sin\theta, 0 < \theta < \pi, 0 < \phi < 2\pi.$$

Ducharme Met al. [3] studied the coverage probabilities of confidence cones with mean vector as the measure of orientation, C_{B1} and C_{RS}, for the von Mises-Fisher distribution.

Each cone was obtained from 200 bootstrap replications. It was checked if they contained $\mu(F)$. This was repeated 1000 times in order to get an estimate of coverage probability. The results are shown in Table 1. The confidence cones with eigenvector as the measure of orientation, C_{B2} and C_{AS}, perform better than the confidence cones with mean vector as the measure of orientation for small sample size and k greater than or equal to 3. As sample size increases, the results of C_{B2} and C_{AS} are similar to C_{B1} and C_{RS}. Bootstrap confidence cones perform well for the increasing sample size. Our bootstrap confidence cone, C_{B2}, performs very well for sample size greater than 20 and k greater than or equal to 3.

n	10			20			50		
k	0.1	1.0	3.0	0.1	1.0	3.0	0.1	1.0	3.0
C_{B1}	.575	.871	.896	.613	.907	.934	.663	.934	.942
C_{DB}	.674	.877	.900	.711	.914	.922	.730	.930	.944
C_{B2}	.625	.800	.902	.635	.908	.942	,668	.931	.946
C_{AS}	.685	.806	.906	.712	.910	.935	.728	.932	.941

Table 1: Coverage probabilities of the confidence cones ($\alpha = 0.05$) based on n observations from the von Mises-Fisher distribution $M(k,(0,0,1))$ with 200 times replications and 1000 times trials.

n	10			20			50		
k	0.1	1.0	3.0	0.1	1.0	3.0	0.1	1.0	3.0
C_{B2}	.637	.796	.905	.690	.897	.941	.687	.934	.949
C_{AS}	.710	.873	.909	.772	.904	.931	.808	.932	.945

Table 2: Coverage probabilities of the confidence cones ($\alpha = 0.05$) based on n observations from the Dimorth-Watson distribution $D(k,(0,0,1))$ with 200 times replications and 1000 times trials.

For the second simulation, we took pseudo-random samples from the Dimorth-Watson distribution by taking special values $\mu(F) = (0,0,1)$ and $k = 0.1$, 1 and 3; $f(x) = cexp(kx_3^2)$ where c is the normalizing constant.

$$f(\theta,\phi) = c(exp(kcos^2\theta))sin\theta, 0 < \theta < \pi, 0 < \phi < 2\pi.$$

The confidence cones C_{B2} and C_{AS} were computed. Each cone was obtained from 200 bootstrap replications. It was checked if they contained $\mu(F)$. This was repeated 1000 times in order to get an estimate of coverage probability. The results of this simulation are shown in Table 2. For small vales of k, bootstrap confidence cone is seen to be unreliable. Our bootstrap confidence confidence cone, C_{B2}, performs well for the increasing sample size and k greater than or equal to 3.

5. Conclusion

From the comparison, we can say that the bootstrap confidence cone for the eigenvector performs quite well. Especially from the comparison of the coverage probabilities of Ducharme's bootstrap confidence cones for the mean vector to those of our bootstrap confidence cone in the case of low concentration($k = 0.1$), we can safely say that the eigenvector is more stable than the mean vector as a measure of orientation for the spherical distributions though the theoretical background for this fact needs further investigated. This outlier-resistant nature of eigenvector makes another case for preferring the eigenvector representation over the mean vector representation in addition to this wider applicability. Hence overall performance of the bootstrap confidence cone for the eigenvector is quite satisfactory. The last but not the least reason for recommending this confidence cone is the automatic nature of construction which is the characteristic of all bootstrap methods.

In conclusion, the eigenvector of the second moment matrix is adequate as a measure of orientation for the spherical distribution and the bootstrap confidence cone for the eigenvector is a reliable companion. Notice that, however, we have only suggested a simple bootstrap method which does not have prior information. Shin [6,7] deals with more realistic bootstrap where the procedure uses prior information and further extends the bootstrap method to the second level in order to get more accuracy of coverage probability.

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