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Estimation in Autoregressive Process with Non-negative Innovations

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ABSTRACT

In this paper, we obtain the natural estimators of the coefficient parameters and propose strongly consistent estimators of the parameter in the autoregressive model of order three with non-negative innovations. It is shown that the natural estimators are also strongly consistent for the parameters. We also compare the proposed estimators with the natural estimators and the least square estimators via Monte Carlo simulation studies.

1. Introduction

In the time series analysis, the most crucial steps are to identify and to build a model based on available data. In this step we usually assume that the innovations are Gaussian random variables with mean zero and finite constant variance. But in the real situations, it is known that the time series processes commonly have non-Gaussian innovations (Nelson and Granger(1979)).

In the autoregressive process of order one with the non-negative innovations, Bell and Smith(1986) studied the moment structure of the process and proposed a consistent estimator of the coefficient parameter and called it the natural estimator. they also proved that the natural estimator is the maximum likelihood estimator of the parameter when the distribution of the innovations follows an one parameter exponential distribution. Moreover, Anděl(1988) derived the distribution of the

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natural estimator. Furthermore Anděl(1989) also derived the autoregressive process of order two with the non-negative innovation.

Here we consider the natural estimators of the coefficient parameters in the autoregressive model of order three with non-negative innovations, and we find a properties of these estimators. Also we propose another strongly consistent estimators of the coefficient parameters in the same model, and we compare the proposed strongly consistent estimators with the natural estimators and the least square estimators in terms of bias and mean square errors via Monte Carlo simulation studies.

In Section 2, we obtain the natural estimators of the coefficient parameters in the autoregressive process of order three with the non-negative innovations and prove that these are strongly consistent.

In Section 3, we propose another strongly consistent estimators of the coefficient parameters in the same model used in Section 2. we also show that the proposed estimators are the maximum likelihood estimators when the innovations are exponential distribution.

In Section 4, we compare the proposed estimators with the natural estimator and the least square estimators in terms of bias and mean square errors through the Monte Carlo simulation method.

2. Natural Estimators and Their Properties

Consider the stationary autoregressive process of order three defined by

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + b_3 X_{t-3} + \varepsilon_t, \tag{1}$$

where t = 4, 5, ... ,n. It is assumed that the innovations ε_t be positive independent identically distributed random variables with a distribution function F and have finite unknown constant variance. We also assume that the coefficient parameters in the model (1) are non-negative and satisfy the stationary condition.

Let $X_1, \cdots, X_n$ be the realizations from the model (1). Then one easily can obtain the followings

$$\frac{X_t}{X_{t-1}} = b_1 + \frac{b_2 X_{t-2} + b_3 X_{t-3} + \varepsilon_t}{X_{t-1}}. \tag{2}$$

Intuitively, if the size of realizations is sufficiently large, then one can expect with a high probability that at least one t exists such that X_{t-2}, X_{t-3}, and ε_t are small and X_{t-1} is very large. That is, one can expect with a high probability that there exist t such that left hand side of equation (2) approximately equal to b_1. In fact, the last term in the equation (2) can be represented as

$$C(\lambda_k, \varepsilon_{t-k}, \varepsilon_{t-1}) \equiv \sum_{k=0}^{\infty} \frac{\lambda_k \varepsilon_{t-k}}{\varepsilon_{t-1}}, \tag{3}$$

where $\lambda_0 = 1$, $\lambda_1 = 0$, $\lambda_k = b_2 c_{k-2} + b_3 c_{k-1}$ for $k \geq 2$ and c_j is a function of b_i's. Using the method of proof in Theorem A.1 in Anděl(1989), it is not difficult to show that minimum of (3) almost surely converges to zero as n approach to infinity. Thus we obtain the following result.

Theorem 2.1. Let $X_1, X_2, \cdots, X_n$ be a realizations from the stationary autoregressive process (1) and define

$$b_1^+ = \min_{4 \leq t \leq n} \left(\frac{X_t}{X_{t-1}} \right).$$

Then b_1^+ is a strongly consistent estimator of b_1.

Similarly, from (1), we also obtain the followings

$$\frac{X_t}{X_{t-2}} = b_2 + \frac{b_1 X_{t-1} + b_3 X_{t-3} + \varepsilon_t}{X_{t-2}}. \tag{4}$$

In (4), if X_{t-1} is substituted by $b_1 X_{t-2} + b_2 X_{t-3} + b_3 X_{t-4} + \varepsilon_{t-1}$, then

$$\frac{X_t}{X_{t-2}} = b_2 + b_1^2 + \frac{(b_1 b_2 + b_3) X_{t-3} + b_1 b_3 X_{t-4} + b_1 \varepsilon_{t-1} + \varepsilon_t}{X_{t-2}}. \tag{5}$$

The last term in the equation (5) is C(λ_k, ε_{t-k}, ε_{t-2}) with $\lambda_0 = 1$, $\lambda_1 = b_1$, $\lambda_2 = 0$, $\lambda_k = (b_1 b_2 + b_3) c_{k-3} + b_1 b_3 c_{k-2}$ for $k \geq 3$ and c_j is a function of b_i. Thus it can be shown that minimum of C(λ_k, ε_{t-k}, ε_{t-2}) almost surely converges to zero as n approach to infinity. Thus the following theorem holds.

Theorem 2.2. Under the conditions as those in Theorem 2.1, if we define

$$b_2^+ = \min_{4 \leq t \leq n} \left(\frac{X_t}{X_{t-2}} \right).$$

Then b_2^+ is a strongly consistent estimator of $b_2 + b_1^2$.

From Theorem 2.1 and Theorem 2.2, the following corollary can be obtained.

Corollary 2.1. Under the same conditions as those in Theorem 2.2, $b_2^+ - {b_1^+}^2$ is a strongly consistent estimator of b_2.

Finally, from (1) the following equation holds.

$$\frac{X_t}{X_{t-3}} = b_3 + \frac{b_1 X_{t-1} + b_2 X_{t-2} + \varepsilon_t}{X_{t-3}}. \tag{6}$$

In (6), if X_{t-1} and X_{t-2} are substituted by $b_1 X_{t-2} + b_2 X_{t-3} + b_3 X_{t-4} + \varepsilon_{t-1}$ and $b_1 X_{t-3} + b_2 X_{t-4} + b_3 X_{t-5} + \varepsilon_{t-2}$ respectively, then we obtain

$$\begin{aligned}\frac{X_t}{X_{t-3}} =& b_3 + 2b_1 b_2 + b_1^3 + \frac{(b_1^2 b_2 + b_1 b_3 + b_2^2)X_{t-4}}{X_{t-3}} \\ &+ \frac{(b_1^2 b_3 + b_2 b_3)X_{t-5} + (b_1^2 b_2)\varepsilon_{t-2} + b_1 \varepsilon_{t-1} + \varepsilon_t}{X_{t-3}}.\end{aligned} \tag{7}$$

The last term in (7) is also $C(\lambda_k,\ \varepsilon_{t-k},\ \varepsilon_{t-3})$ with $\lambda_0 = 1$, $\lambda_1 = 0$, $\lambda_2 = b_1^2 + b_2$, $\lambda_k = (b_1^2 b_2 + b_1 b_3 + b_2^2)c_{k-4} + (b_1^2 b_3 + b_2 b_3)c_{k-3}$ for $k \geq 4$ and c_j is a function of b_i.

Similary, it can be shown that minimum of $C(\lambda_k,\ \varepsilon_{t-k},\ \varepsilon_{t-3})$ almost surely converges to zero as n approaches to infinity. Thus the following theorem holds.

Theorem 2.3. Under the same conditions as those in Theorem 2.2, let

$$b_3^+ = \min_{3 \leq t \leq n} \left(\frac{X_t}{X_{t-3}} \right).$$

Then b_3^+ is a strongly consistent estimator of $b_3 + 2b_1 b_2 + b_1^3$.

From Theorem 2.1, Theorem 2.2 and Theorem 2.3, one can easily obtain the following result.

Corollary 2.2. Under the same conditions as those in Theorem 2.3, $b_3^+ - 2b_1^+ b_2^+ - {b_1^+}^3$ is a strongly consistent estimator of b_3.

3. Proposed Estimators

In this section, we propose another strongly consistent estimators for the coefficient parameters, respectively, in the non-negative autoregressive process given in the previous section, which are asymptotic maximum likelihood estimators if the underlying distribution is exponential one.

In fact, the proposed estimators are the values of the parameters b_1, b_2 and b_3 in the given model which maximize $b_1 + b_2 + b_3$ subject to the conditions $X_t - b_1X_{t-1} - b_2X_{t-2} - b_3X_{t-3} \geq 0$ for t=4, 5, ... ,n. we also show that the proposed estimators are strongly consistent for the parameters in the model. At first, we show that those are maximum likelihood setimators of the parameters in the case when the innovations are independent exponential distributions. Under the assumption of exponential distributions with single scale parameter λ, the likelihood function , given $X_1 = x_1$, $X_2 = x_2$, and $X_3 = x_3$, is

$$L = \lambda^{-n+3} exp\left(-\sum_{t=4}^{n} \frac{x_t - b_1x_{t-1} - b_2x_{t-2} - b_3x_{t-3}}{\lambda}\right)$$

for $x_t - b_1x_{t-1} - b_2x_{t-2} - b_3x_{t-3} \geq 0,\ t = 4, 5, \ldots, n$. Thus the maximum likelihood estimators of the parameters b_1, b_2 and b_3 are the values of b_1, b_2 and b_3 which maximize the followings

$$b_1 \sum_{t=4}^{n} x_{t-1} + b_2 \sum_{t=4}^{n} x_{t-2} + b_3 \sum_{t=4}^{n} x_{t-3} \tag{8}$$

subject to

$$x_t - b_1x_{t-1} - b_2x_{t-2} - b_3x_{t-3} \geq 0, \tag{9}$$

for t=4, 5, ... ,n. In the time series analysis, generally the realization size n is usually large. Thus the terms $\sum_{t=4}^{n} x_{t-1}, \sum_{t=4}^{n} x_{t-2}$ and $\sum_{t=4}^{n} x_{t-3}$ in (8) probably do not differ very much. Therefore, the values of b_1, b_2 and b_3 which maximize $b_1 + b_2 + b_3$ under the conditions (9) are the asymptotic maximum likelihood estimators of the parameters in the given model.

However, It is not assumed that the innovations are independent exponentially distributed ramdom variables. In this paper, we only assume that they are independent identical non-negative distributions with finite unknown constant variance.

But it is interesting that estimators obtained by maximizing $b_1 + b_2 + b_3$ under the condition (9) and the above assumptions for innovations have good statistical properties in the sense of bias, mean square error and consistency. we can lead to the following result.

Theorem 3.1. Let b_1^*, b_2^* and b_3^* be values of coefficient parameters b_1, b_2 and b_3,respectively, in the given model (1) which maximize $b_1 + b_2 + b_3$ subject to the conditions (9). Then b_1^*, b_2^* and b_3^* are strongly consistent estimators of b_1, b_2 and b_3 respectively.

Proof. First, define a parametic space with the condition (9) as follows

$$\begin{aligned} M_n = \{ \ (\beta_1, \beta_2, \beta_3) : \beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_3 \geq 0, \\ X_t - \beta_1 X_{t-1} - \beta_2 X_{t-2} - \beta_3 X_{t-3} \geq 0 \\ for \quad t = 4, 5, \ldots, n\}. \end{aligned}$$

It is clear that $M_4 \supset M_5 \supset M_6 \supset \cdots$. We want to find a set M such that M_n converges to M almost surely as n approaches to infinity and to find the values of β_1, β_2 and β_3 which maximize $\beta_1 + \beta_2 + \beta_3$ in M. We consider a plane Z given by $X_t - \beta_1 X_{t-1} - \beta_2 X_{t-2} - \beta_3 X_{t-3} = 0$ (See Figure A). If the values of β_1 and β_2 are zero on the plane Z, then the value of β_3 is X_t / X_{t-3}. also if the values of β_1 and β_3 are zero on the plane Z, then the value of β_2 is X_t / X_{t-2}, and also if the values of β_2 and β_3 are zero on the plane Z, then the value of β_1 is X_t / X_{t-1}.

Step 1 : By Theorem 2.1., since b_1^+ converges to b_1 almost surely as n approaches to infinity, there exist indices t_n such that $X_{t_n} / X_{t_n - 1}$ converges to b_1. Using (2), we obtain $(b_2 X_{t_n - 2} + b_3 X_{t_n - 3} + \varepsilon_{t_n}) / X_{t_n - 1}$ approaches to zero as n approach to infinity. Now, X_{t_n} and ε_{t_n} are positive random variables and $b_2 \geq 0$, $b_3 \geq 0$. Therefore $X_{t_n - 2} / X_{t_n - 1}$, $X_{t_n - 3} / X_{t_n - 1}$ and $\varepsilon_{t_n} / X_{t_n - 1}$ converges to zero. Since

$$\frac{X_{t_n}}{X_{t_n - 2}} = b_2 + \frac{b_1 X_{t_n - 1} + b_3 X_{t_n - 3} + \varepsilon_{t_n}}{X_{t_n - 2}},$$

$X_{t_n} / X_{t_n - 2}$ diverges to infinity. Also since

$$\frac{X_{t_n}}{X_{t_n - 3}} = b_3 + \frac{b_1 X_{t_n - 1} + b_2 X_{t_n - 2} + \varepsilon_{t_n}}{X_{t_n - 3}},$$

$X_{t_n} / X_{t_n - 3}$ diverges to infinity.

In this case, Z approaches the plane which passes through the point b_1 on β_1-axis and parallels to the β_2-axis and to the β_3-axis.

Step 2 : By Theorem 2.2., since b_2^+ converges to $b_2 + b_1^2$ almost surely as n approaches to infinity, there exist indices s_n such that $\{(b_1 b_2 + b_3)X_{s_n-3} + b_1 b_2 X_{s_n-4} + b_1 \varepsilon_{s_n-1} + \varepsilon_{s_n}\}/X_{s_n-2}$ converge to zero. Since X_{s_n} and ε_{s_n} are positive random variables and $b_1 \geq 0,\ b_2 \geq 0,\ b_3 \geq 0,\ X_{s_n-3}/X_{s_n-2},\ X_{s_n-4}/X_{s_n-2},\ \varepsilon_{s_n-1}/X_{s_n-2}$ and $\varepsilon_{s_n}/X_{s_n-2}$ converge to zero as n approaches to infinity. From the equation (2),

$$\begin{aligned}\frac{X_{s_n}}{X_{s_n-1}} =& b_1 + \frac{b_2}{b_1} + \frac{(b_3 - b_1^{-1}b_2^2)X_{s_n-3}}{X_{s_n-1}} \\ &+ \frac{-b_1^{-1}b_2 b_3 X_{s_n-4} - b_1^{-1}b_2\varepsilon_{s_n-1} + \varepsilon_{s_n}}{X_{s_n-1}}.\end{aligned}$$

In fact,

$$\begin{aligned}&\left|\frac{(b_3 - b_1^{-1}b_2^2)X_{s_n-3} - b_1^{-1}b_2 b_3 X_{s_n-4} - b_1^{-1}b_2\varepsilon_{s_n-1} + \varepsilon_{s_n}}{b_1 X_{s_n-2} + b_2 X_{s_n-3} + b_3 X_{s_n-4} + \varepsilon_{s_n-1}}\right| \\ \leq& \frac{(b_3 - b_1^{-1}b_2^2)X_{s_n-3}}{b_1 X_{s_n-2}} + \frac{b_1^{-1}b_2 b_3 X_{s_n-4}}{b_1 X_{s_n-2}} + \frac{b_1^{-1}b_2\varepsilon_{s_n-1}}{b_1 X_{s_n-2}} + \frac{\varepsilon_{s_n}}{b_1 X_{s_n-2}}.\end{aligned}$$

The right hand side of above inequality converges to zero as n approaches to infinity. Therefore, X_{s_n}/X_{s_n-1} converges to $b_1 + b_2/b_1$ as n approaches to infinity. Also from the equation (6),

$$\frac{X_{s_n}}{X_{s_n-3}} = b_3 + b_1 b_2 + \frac{(b_1^2 + b_2)X_{s_n-2} + b_1 b_3 X_{s_n-4} + b_1\varepsilon_{s_n-1} + \varepsilon_{s_n}}{X_{s_n-3}}. \tag{10}$$

As the term X_{s_n-2}/X_{s_n-3} in the equation (10) diverges to infinity, X_{s_n}/X_{s_n-3} diverges to infinity as n approaches to infinity.

In this case one can see that the plane Z approaches the plane which passes through the point $b_1 + b_2/b_1$ on the β_1-axis and $b_2 + b_1^2$ on β_2-axis and parallels to the β_3-axis.

Step 3 : By Theorem 2.3., since b_3^+ converge to $b_3 + 2b_1 b_2 + b_1^3$ almost surely as n approaches to infinity, there exist indices v_n such that, when n approaches to infinity,

$$\frac{(b_1^2b_2+b_1b_2+b_2^3)X_{v_n-4}+(b_1^2b_3+b_2b_3)X_{v_n-5}}{X_{v_n-3}} + \frac{(b_1^2+b_2)\varepsilon_{v_n-2}+b_1\varepsilon_{v_n-1}+\varepsilon_{v_n}}{X_{v_n-3}}$$

approaches to zero. Since X_{v_n-3}, X_{v_n-4}, X_{v_n-5}, ε_{v_n}, ε_{v_n-1} and ε_{v_n-2} are positive random variables, $b_1 \geq 0$, $b_2 \geq 0$ and $b_3 \geq 0$, we obtain that X_{v_n-4}/X_{v_n-3}, X_{v_n-5}/X_{v_n-3}, $\varepsilon_{v_n-2}/X_{v_n-3}$, $\varepsilon_{v_n-1}/X_{v_n-3}$ and $\varepsilon_{v_n}/X_{v_n-3}$ converge to zero as n approaches to infinity. Also from the equation (2),

$$\begin{aligned}\frac{X_{v_n}}{X_{v_n-1}} =& b_1+\frac{b_2}{b_1}+\frac{b_1^{-1}(b_3-b_1^{-1}b_2^2)X_{v_n-2}+(b_1^{-2}b_2^3-2b_1^{-1}b_2b_3)X_{v_n-4}}{X_{v_n-1}}\\ &+\frac{-b_1^{-1}b_3(b_3-b_1^{-1}b_2^2)X_{v_n-5}-b_1^{-1}(b_3-b_1^{-1}b_2^2)\varepsilon_{v_n-2}}{X_{v_n-1}}\\ &-\frac{b_1^{-1}b_2\varepsilon_{v_n-1}+\varepsilon_{v_n}}{X_{v_n-1}}.\end{aligned}$$

Now, we obtain that, when n approaches to infinity, $\{b_1^{-1}(b_3-b_1^{-1}b_2^2)X_{v_n-2}\}/X_{v_n-1}$ converge to $(b_3-b_1^{-1}b_2^2)/(b_1^2+b_2)$, and since $X_{v_n-1} = (b_1^2+b_2)X_{v_n-3}+(b_1b_2+b_3)X_{v_n-4}+b_1b_3X_{v_n-5}+b_1\varepsilon_{v_n-2}+\varepsilon_{v_n-1}$, we obtain the following result.

$$\begin{aligned}&\left|\frac{(b_1^{-2}b_2^3-2b_1^{-1}b_2b_3)X_{v_n-4}-b_1^{-1}b_3(b_3-b_1^{-1}b_2^2)X_{v_n-5}}{X_{v_n-1}}\right.\\ &\left.+\frac{-b_1^{-1}(b_3-b_1^{-1}b_2^2)\varepsilon_{v_n-2}-b_1^{-1}b_2\varepsilon_{v_n-1}+\varepsilon_{v_n}}{X_{v_n-1}}\right|\\ \leq&\frac{(b_1^{-2}b_2^3-2b_1^{-1}b_2b_3)X_{v_n-4}}{(b_1^2+b_2)X_{v_n-3}}+\frac{b_1^{-1}b_2(b_3-b_1^{-1}b_2^2)X_{v_n-5}}{(b_1^2+b_2)X_{v_n-3}}\\ &+\frac{b_1^{-1}(b_3-b_1^{-1}b_2^2)\varepsilon_{v_n-2}}{(b_1^2+b_2)X_{v_n-3}}+\frac{b_1^{-1}b_2\varepsilon_{v_n-1}}{(b_1^2+b_2)X_{v_n-3}}+\frac{\varepsilon_{v_n}}{(b_1^2+b_2)X_{v_n-3}}.\end{aligned}$$

The right hand side of above inequality converges to zero as n approaches to infinity. Thus, X_{v_n}/X_{v_n-1} converges to $b_1+b_2/b_1+(b_3-b_1^{-1}b_2^2)/(b_1^2+b_2)$, as n approaches to infinity. Also, from the equation (4), one can lead to

$$\begin{aligned}\frac{X_{v_n}}{X_{v_n-2}} =& 2b_2 + b_1^2 + \frac{b_3}{b_1} + \frac{(b_1b_3 - b_2^2 - b_1^{-1}b_2b_3)X_{v_n-4}}{X_{v_n-2}} \\ &+ \frac{-(b_2b_3 + b_1^{-1}b_3^2)X_{v_n-5} - (b_2 + b_1^{-1}b_3)\varepsilon_{v_n-2}}{X_{v_n-2}} \\ &+ \frac{b_1\varepsilon_{v_n-1} + \varepsilon_{v_n}}{X_{v_n-2}}.\end{aligned}$$

Now, we obtain the following inequality

$$\begin{aligned}&\left| \frac{(b_1b_3 - b_2^2 - b_1^{-1}b_2b_3)X_{v_n-4} - (b_2b_3 + b_1^{-1}b_3^2)X_{v_n-5}}{b_1X_{v_n-3} + b_2X_{v_n-4} + b_3X_{v_n-5} + \varepsilon_{v_n-2}} \right. \\ &\left. + \frac{-(b_2 + b_1^{-1}b_3)\varepsilon_{v_n-2} + b_1\varepsilon_{v_n-1} + \varepsilon_{v_n}}{b_1X_{v_n-3} + b_2X_{v_n-4} + b_3X_{v_n-5} + \varepsilon_{v_n-2}} \right| \\ \le& \frac{(b_1b_3 - b_2^2 - b_1^{-1}b_2b_3)X_{v_n-4}}{b_1X_{v_n-3}} + \frac{(b_2b_3 + b_1^{-1}b_3^2)X_{v_n-5}}{b_1X_{v_n-3}} \\ &+ \frac{(b_2 + b_1^{-1}b_3)\varepsilon_{v_n-2}}{b_1X_{v_n-3}} + \frac{b_1\varepsilon_{v_n-1}}{b_1X_{v_n-3}} + \frac{\varepsilon_{v_n}}{b_1X_{v_n-3}}.\end{aligned}$$

It can be also shown that the right hand side of above inequality converges to zero as n approaches to infinity. Thus X_{v_n}/X_{v_n-2} approaches to $2b_2 + b_1^2 + b_3/b_1$ as n approaches to infinity.

In this case, one can see that the plane Z approaches the plane which passes through the point $b_1 + b_2/b_1 + (b_3 - b_1^{-1}b_2^2)/(b_1^2 + b_2)$ on the β_1-axis and the point $2b_2 + b_1^2 + b_3/b_1$ on the β_2-axis and the point $b_3 + 2b_1b_2 + b_1^3$ on the β_3-axis.

From the Step 1, Step 2 and Step 3, one can get

$$\begin{aligned}M = \{ (\beta_1, \beta_2, \beta_3) \mid & 0 \le \beta_1 \le b_1, 0 \le \beta_2 \le b_2 + b_1^2, \\ & 0 \le \beta_3 \le b_3 + 2b_1b_2 + b_1^3, \\ & \beta_3 \le b_3 + b_1^3 + 2b_1b_2 - (b_1^2 + b_2)\beta_1 - b_1\beta_2, \\ & \beta_2 \le b_2 + b_1^2 - b_1\beta_1 \}.\end{aligned}$$

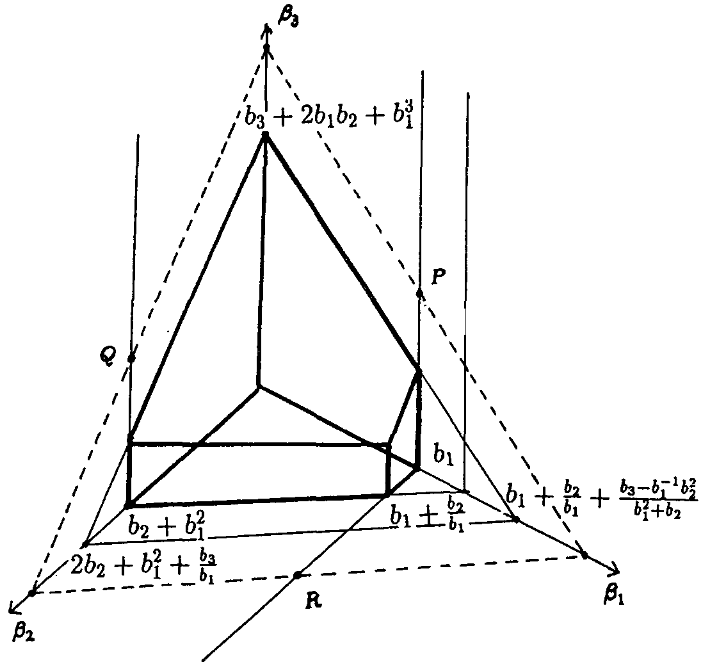

Figure A

Moreover, it can be proved that none of the planes for all $t \geq 4$ can pass through a point located in the interior of M. In the Figure A, the points P, Q and R are represented as follows.

$$
\begin{aligned}
P &= -\frac{X_{t-1}}{X_{t-3}}\beta_1 + \frac{X_t}{X_{t-3}} \\
&= b_3 + b_1 b_2 + \frac{b_2^2 X_{t-4} + b_2 b_3 X_{t-5} + b_2 \varepsilon_{t-2} + \varepsilon_t}{X_{t-3}} \\
&\geq b_3 + b_1 b_2.
\end{aligned}
$$

$$
\begin{aligned}
Q &= -\frac{X_{t-2}}{X_{t-3}}\beta_2 + \frac{X_t}{X_{t-3}} \\
&= b_3 + b_1 b_2 + \frac{2b_1^2 X_{t-2} + b_1 b_3 X_{t-4} + b_1 \varepsilon_{t-1} + \varepsilon_t}{X_{t-3}} \\
&\geq b_3 + b_1 b_2.
\end{aligned}
$$

$$\begin{aligned} R &= -\frac{X_{t-1}}{X_{t-2}}\beta_1 + \frac{X_t}{X_{t-2}} \\ &= b_2 + \frac{b_3 X_{t-3} + \varepsilon_t}{X_{t-2}} \\ &\geq b_2. \end{aligned}$$

Therefore, the points P, Q and R are not located in the interior of M.

Finally, It needs to find the values of β_1, β_2 and β_3 which maximize $\beta_1 + \beta_2 + \beta_3$ in the space M. Since

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 &= \beta_1 + \beta_2 + (b_3 + b_1^3 + 2b_1 b_2) - (b_1^2 + b_2)\beta_1 - b_1\beta_2 \\ &= b_3 + b_1^3 + 2b_1 b_2 + b_1^2 + [1 - (b_1 + b_2)]\beta_1, \end{aligned}$$

the maximum value of β_1 in the space M maximizes $\beta_1 + \beta_2 + \beta_3$. Hence β_1, β_2 and β_3 should approach to the points b_1, b_2 and b_3, respectively as n approaches to infinity.

Remark. The estimators b_1^*, b_2^* and b_3^* are easily computed by the simplex or dual simplex method.

4. Monte Carlo Simulation

In this section, we investigate the performances of the natural estimators, the proposed estimators and the least square estimators via Monte Carlo simulation in terms of the bias and mean square error(MSE). We denote the least square estimators(LSE) of b_1, b_2 and b_3 by b_1^o, b_2^o and b_3^o, respectively.

In the simulation study, the true values of the parameters in the model (1) are considered as $b_1 = 0.5$, $b_2 = 0.3$ and $b_3 = 0.1$, and three distributions of the white noise ϵ_i are investigated. We generate pseudorandom samples of size 30, 50, 70 and 100 (We use GGUBS(uniform), GGEXN(exponential) and GGNLG(lognormal) in the IMSL), and we evaluate the biases and MSE's for the natural estimators, the proposed estimators and the least square estimators of the parameters.

Table 1 shows that the biases and MSE's for the estimators when the innovations ϵ_i are exponential distribution withthe scale parameter one.

Table 2 show that when the innovations ϵ_i are uniform distribution with parameters zero and one, that is, $\epsilon_i \sim$ UNIF(0,1). Finally the biases and MSE's for the estimators when the innovations ϵ_i are lognormal distribution with parameters zero and one are given in Table 3.

In the case $\epsilon_i \sim$ EXP(1), from the Table 1 one can see the following facts.

(1) For $n = 30$, the estimators b_1^+, b_2^+ and b_3^+ have very large biases and MSE's. The bias is not negligible even for extremely large n. That is, for $n = 50$, 70, 100, the bias for the estimators b_1^+, b_2^+ and b_3^+ are larger than other estimators.

(2) The estimators b_1^*, b_2^* and b_3^* have smaller biases than those of the LSE's. In terms of bias, the estimators b_1^*, b_2^* and b_3^* are better than other estimators. In terms of MSE, the estimators b_1^*, b_2^* and b_3^* have larger than the LSE for the size $n = 30$, 50 and 70. But for $n = 100$, b_1^*, b_2^* and b_3^* have smaller than the LSE.

In the case $\epsilon_i \sim$ UNIF(0,1), from the Table 2 one can see the following facts.

(1) For $n = 30$, 50, 70 and 100, the estimators b_1^+, b_2^+ and b_3^+ have large biases and MSE's. From Table 1, the bias in not negligible even for extremely large n. Also, the MSE is larger than other estimators for large n.

(2) For $n = 30$, 50, 70 and 100. The estimators b_1^*, b_2^* and b_3^* have smaller biases than the LSE's. Thus, in terms of bias, the estimators b_1^*, b_2^* and b_3^* are better than other estimators. In terms of MSE, the estimator b_1^* have larger MSE than the LSE for the size $n = 30$ and 50. But for the size $n = 70$ and 100, the estimator b_1^* have smaller than the LSE. Also the estimators b_2^* and b_3^* have larger than the LSE for the size $n = 30$, 50 and 70, but for the size $n = 100$, the estimators b_2^* and b_3^* have smaller than the LSE.

For the case $\log \epsilon_i \sim$ N(0,1), we also obtain the analogous results as those of $\epsilon_i \sim$ EXP(1) or $\epsilon_i \sim$ UNIF(0,1) from the Table 3.

Table 1. Comparisons of the Bias and MSE for the Natual Estimators, the Proposed Estimators and the Least Square Estimators when $\varepsilon_t \sim$ EXP(1)

n		Natural		Proposed		LS	
		Bias	MSE	Bias	MSE	Bias	MSE
	b_1	.3325	.1118	.0092	.0120	.0494	.0025
30	b_2	-.1635	.0291	.0035	.0093	.0451	.0015
	b_3	-.1569	.0285	.0231	.0077	.0467	.0023
	b_1	.3183	.1024	.0082	.0040	.0459	.0017
50	b_2	-.1525	.0252	.0029	.0035	.0362	.0014
	b_3	.1501	.0257	-.0044	.0021	.0339	.0012
	b_1	.3134	.0992	.0058	.0020	.0409	.0016
70	b_2	-.1482	.0235	.0020	.0019	.0281	.0009
	b_3	-.1452	.0242	.0009	.0016	.0321	.0011
	b_1	.3059	.0945	.0044	.0009	.0399	.0014
100	b_2	-.1435	.0223	.0002	.0009	.0292	.0009
	b_3	-.1388	.0220	-.0005	.0007	.0298	.0009

Table 2. Comparisons of the Bias and MSE for the Natual Estimators, the Proposed Estimators and the Least Square Estimators when $\varepsilon_t \sim$ UNIF(0,1)

n		Natural		Proposed		LS	
		Bias	MSE	Bias	MSE	Bias	MSE
	b_1	.3979	.1584	.0647	.0216	.0676	.0023
30	b_2	-.2323	.0545	.0158	.0249	-.0378	.0023
	b_3	-.0995	.0097	.0513	.0198	.0548	.0033
	b_1	.3887	.1513	.0333	.0127	.0617	.0046
50	b_2	-.2218	.0497	.0057	.0159	-.0055	.0021
	b_3	-.0981	.0103	.0273	.0085	.0343	.0025
	b_1	.3835	.1472	.0256	.0039	.0444	.0040
70	b_2	-.2182	.0481	-.0049	.0091	-.0034	.0017
	b_3	-.0978	.0102	.0163	.0049	.0220	.0020
	b_1	.3752	.1409	.0005	.0028	.0296	.0029
100	b_2	-.2113	.0450	.0026	.0015	.0028	.0015
	b_3	-.0973	.0105	.0131	.0018	.0214	.0019

Table 3. Comparisons of the Bias and MSE for the Natual Estimators, the Proposed Estimators and the Least Square Estimators when $log\varepsilon_t \sim$ N(0,1)

		Natural		Proposed		LS	
n		Bias	MSE	Bias	MSE	Bias	MSE
	b_1	.3098	.0991	.0106	.0101	.0568	.0033
30	b_2	-.1379	.0226	.0031	.0091	.0478	.0019
	b_3	-.0894	.0096	.0311	.0089	.0398	.0017
	b_1	.2922	.0880	.0104	.0040	.0516	.0028
50	b_2	-.1258	.0189	.0020	.0040	.0394	.0019
	b_3	-.0871	.0091	.0043	.0029	.0251	.0007
	b_1	.2761	.0791	.0085	.0023	.0447	.0021
70	b_2	-.1149	.0162	.0018	.0022	.0379	.0016
	b_3	-.0827	.0083	-.0034	.0018	.0195	.0008
	b_1	.2629	.0718	.0004	.0012	.0398	.0016
100	b_2	-.1062	.0138	.0014	.0012	.0360	.0013
	b_3	-.0818	.0079	-.0018	.0011	.0143	.0007

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