SEMIGROUPS AND HIGHER ORDER GENERATORS

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The semigroup theory, the properties of its generators and its applications to the mixed problems and the initial value problems of partial differential equations have been considered by several authors (see for examples [1], [2], [4] and [6]). Here we will show that if the operator $A$ generates a contraction semigroup, then under certain conditions (which will be stated) the operators $A^{2m}$, $A^m$ and $\sum_{k=1}^{m} A^k$ generate analytic semigroups. Some examples of generators will be given to illustrate the results, and higher order equations of evolution will be considered.

1. Introduction

Let $X$ be a Hilbert space and $A$ be a linear operator defined in $X$. Let us consider the equation of evolution

$$\frac{du(t)}{dt} + Au(t) = f(t).$$

(1.1)

It is known [8] that if the operator $A$ is closed densely in $X$, then $A$ generates a contraction semigroup $\{T(t), t \geq 0\}$ of bounded linear operators if and only if $A$ and its adjoint $A^*$ are maximal dissipative. Now some definitions are listed for later use.

Definition 1.1. Let $A$ be linear operator in the Hilbert space $X$, and its domain is assumed to be dense in $X$. The operator $A$ is said to be dissipative if $Re(Au,u) \leq 0$ or equivalently

$$\|(A - \lambda)u\| \geq Re\lambda \|u\|, \text{ for all } u \in D(A) \text{ and } Re\lambda > 0.$$
If $\text{Re}(Au, u) \geq 0$, that is $-A$ is dissipative, $A$ is said to be accretive. A dissipative operator which extends a dissipative operator is called a dissipative extension of $A$. An operator $A$ is said to be maximal dissipative if its only dissipative extension is the operator $A$ itself. Accretive extensions and maximal accretive operators are defined similarly.

**Definition 2.1.** Let $T(t)$ for each $t \in [0, \infty)$ be a bounded linear operator in $X$, $\{T(t)\}$ is called a semigroup of bounded operators if

1. $T(t + s) = T(t)T(s) = T(s)T(t)$, $s, t \geq 0$
2. $T(0) = I$
3. $T(t)$ is strongly continuous in $t \in [0, \infty)$.

It is known [8, p.53] that there exists real numbers $M \geq 1$ and $\beta$ such that

$$
\|T(t)\| \leq Me^{\beta t} \tag{1.2}
$$

for all $t \geq 0$.

A semigroup $\{T(t)\}$ is called a contraction semigroup if it satisfies

$$
\|T(t)\| \leq 1.
$$

When $A$ generates a semigroup satisfies (1.2), we write $A \in G(X, M, \beta)$, and when the semigroup is a contraction semigroup we write $A \in G(X, 1, 0)$.

2. **Higher Order Generators**

Here we assume that $A \in G(X, 1, 0)$. Using the results of [3], [5] and [8] we have the following lemmas.

**Lemma 2.1.** If $A \in G(X, 1, 0)$, then $D(A^{2m})$ is dense in $X$, $m = 1, 2, 3, \ldots$

**Proof.** Since $A \in G(X, 1, 0)$, it follows that $A$ is closed densely defined in $X$, and $A$ and $A^*$ are maximal dissipative. Hence $-A$ is maximal accretive and therefore it is of type ($\frac{\pi}{2}, 1$) (for notations see [8] p.32) from which we can deduce [8] that

$$
D((-A)^{2m}) = D(A^{2m}) \text{ is dense in } X, \text{ where } m = 1, 2, 3, \ldots
$$

**Lemma 2.2.** Let $A \in G(X, 1, 0)$. If $A^*(t) = A(t)$ or $A^*(t) = -A(t)$, then $A^{2m}$ is closed.
Proof. Since \((A^{2m})^* = (A^*)^{2m} = (\pm A)^{2m} = A^{2m}\) it follows that \(A^{2m}\) is self-adjoint. Now \(A^{2m}\) is self-adjoint and densely (from Lemma (2.1)) defined in \(X\), hence [3] it is closed operator in \(X\).

**Theorem 2.1.** Let \(A \in G(X, 1, 0)\), if \(A^* = -A\) then

\[ (-1)^{1+m} A^{2m} \in G(X, 1, 0) \]

Proof. From Lemmas (2.1) and (2.2), the operator \(A^{2m}\) is closed densely defined in \(X\). Now let \(m\) be an odd positive integer, we have

\[
Re((A^{2m} - \gamma)u, u) = Re(A^m u, A^{*m} u) - Re\gamma \|u\|^2 \\
= Re(A^m u, -A^m u) - Re\|u\|^2 \\
= -\|A^m u\|^2 - Re\|u\|^2 \\
\leq -Re\|u\|^2,
\]

and so

\[
\|(A^{2m} - \gamma)u\| \geq -Re((A^{2m} - \gamma) \geq -Re((A^{2m} - \gamma)u, u) \geq Re\|u\|^2
\]

i.e.

\[
\|(A^{2m} - \gamma)u\| \geq Re\|u\|, Re\gamma > 0
\]

therefore \(A^{2m}\) and its adjoint are dissipative, from which we can deduce [8] that they are maximal dissipative in \(X\), and then

\[
A^{2m} \in G(X, 1, 0)
\] (2.2)

Secondly let \(m\) be an even positive integer, we have

\[
Re(A^{2m} + \gamma)u, u) = Re(A^m u, A^{*m} u) + Re\|u\|^2 \\
= Re(A^m u, A^m u) + Re\|u\|^2 \\
= \|A^m u\|^2 + Re\|u\|^2 \\
\geq Re\|u\|^2, Re\gamma > 0
\]

and so

\[
\|(A^{2m} + \gamma)u\| < Re\|u\|,
\]

therefore \(-A^{2m}\) and its adjoint are dissipative, from which we can deduce [8] that they are maximal dissipative, and then

\[
-A^{2m} \in G(X, 1, 0).
\] (2.4)
Combining (2.2) and (2.4) we get

\[-1)^{1+m} A^{2m} \in G(X,1,0),\]  

where \(m = 1, 2, \ldots\)

**Theorem 2.2.** Let \(A \in G(X,1,0).\) If \(A\) is self-adjoint, then

\[-A^{2m} \in G(X,1,0), \text{ and } m = 1, 2, \ldots\]

**Proof.** From Lemmas (2.1) and (2.2), \(A^{2m}\) is closed densely defined in \(X.\) Now since

\[Re((A^{2m} + \gamma)u,u) = Re(A^m u, A^m u) + Re\gamma\|u\|^2\]

\[= Re(A^m u, A^m u) + Re\gamma\|u\|^2\]

\[\geq Re\gamma\|u\|^2, Re\gamma > 0\]

and so

\[\|(A^{2m} + \gamma)u\| \geq Re\gamma\|u\|\]  

(2.6)

It follows that \(-A^{2m}\) and its adjoint are dissipative and they are maximal dissipative in \(X\) (cf. [8]). Hence

\[-A^{2m} \in G(X,1,0), \text{ where } m = 1, 2, 3 \ldots\]

which completes the prove.

**Theorem 2.3.** Let \(A \in G(X,1,0)\) and \(A^* = A.\) If \(A\) has a bounded inverse, then

\[-A^m \in G(X,1,0), \text{ } m = 1, 2, 3, \ldots\]

**Proof.** Since \(A^m\) is self-adjoint and has a bounded inverse, it follows that

\[(A^{2m} u, u) = (A^m u, A^m u) = \|A^m u\|^2 > 0\]

for nonvanishing \(u \in D(A^m).\) So \(A^{2m}\) is strictly positive operator, hence \(A^{2m}\) has a strictly positive square root \(A^m\) (cf.[7]) i.e.

\[(A^m u, u) > 0 \text{ for } u \in D(A^m).\]  

(2.7)

Now since \(A^m\) is the square root of \(A^{2m},\) it follows from Lemmas (2.1) and (2.2) (and Proposition 2.3.1 of [8]) that \(A^m\) is closed densely defined in
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\( X \), with domain \( D = D(A^m) \). From (2.7) \( A^m \) and its adjoint are accretive in \( X \), consequently \(-A^m\) and its adjoint are dissipative. Hence they are maximal dissipative and therefore

\[-A^m \in G(X, 1, 0).\]

**Corollary 2.1.** If the assumptions of Theorem (2.3) are satisfied, then

\[ L = - \sum_{k=1}^{m} A^k \in G(X, 1, 0). \]

**Proof.** From Theorem (2.3), \(-A^m \in G(X, 1, 0)\), so it is of type \((\frac{x}{2}, 1)\) (cf [8]). Also \(A^m\) has a bounded inverse, so (by Proposition 2.3.1 of [8]).

\[ D(a^{\alpha}) \supset D(A^{\beta}), \text{ for } 0 < \alpha < \beta. \]

Now for the operator \( L \), take \( D(L) = D(A^m) \), and since

\[ (Lu, u) = - \sum_{k=1}^{m} (A^k u, u) \leq 0 \]

it follows that \( L \) and \( L^* \) are dissipative, consequently they are maximal dissipative and

\[ L \in G(X, 1, 0). \]

**Corollary 2.2.** If \( B \in G(X, 1, 0) \), then the semigroup generated by \( B \) is analytic. So the semigroups (contraction) in Theorems (2.1), (2.2) and (2.3) and Corollary (2.1) are analytic.

**Proof.** If \( B \in G(X, 1, 0) \), then \(-B\) and its adjoint are maximal accretive, from which we deduce [8] that \(-B\) is of type \((\frac{x}{2}, 1)\). So (Theorem 3.3.1 of [8]) the operator \( B \) generates an analytic semigroup. From the uniqueness of the semigroup generated by \( B \), we get the result.

**3. Applications**

Consider the equation of evolution

\[
\frac{du(t)}{dt} - \sum_{k=1}^{m} A^k u(t) = f(t) \quad (3.1)
\]
with the initial data
\[ u(0) = u_0. \quad (3.2) \]

From our results here and as in the case of first order generators, we can easily prove the following theorem.

**Theorem 3.1.** Let \( A \) satisfy the assumptions of Theorem (2.3). If \( u_0 \in X \) and \( f(t) \in C(X, I) \), then the initial value problem (3.1) and (3.2) has a unique solution \( u(t) \in C(X, I) \), given by

\[ u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds, \quad (3.3) \]

with the following properties (for \( t > 0 \)).

1. \( \frac{du(t)}{dt} \in C(X, I) \), and
\[ \| \frac{du(t)}{dt} \| \leq \frac{c}{t} \| u_0 \| + \| f(t) \| + \frac{c}{t} \int_0^t \| f(s) \| ds \quad (3.4) \]

2. \( u(t) \in D(A^m) \), and
\[ \| \sum_{k=1}^m A^k u(t) \| \leq \frac{c}{t} \| u_0 \| + c \int_0^t \| f(s) \| ds \quad (3.5) \]

where \( \{ T(t) \} \) is the semigroup (analytic) generated by \( - \sum_{k=1}^m A^k \), and \( c \) is a positive constant.

**Examples of generators**

1. Let \( \omega \) be a bounded open subset of \( R^n \), with boundary \( \partial \omega \), and define the operator \( A \) as \( D(A) = \{ u \in L_2(\omega) : \Delta u \in L_2(\omega), u|_{\partial \omega} = 0 \} \) and \( Au = -\Delta u \). It is proved in [1] that \( A \in G(X, 1, 0) \) and the operator \( A \) has a bounded inverse. So it satisfies the assumptions of Theorem (2.3). Then the operator \( -\sum_{k=1}^m (-\Delta)^k \) generates an analytic semigroup. Now consider the mixed problem

\[ \frac{\partial u(x, t)}{\partial t} - \sum_{k=1}^m (-\Delta)^k u(x, t) = f(x, t), x \in \omega, t > 0 \quad (3.6) \]

\[ u(x, t) = 0, x \in \partial \omega, t > 0 \quad (3.7) \]

\[ u(x, 0) = u_0(x), x \in \omega \quad (3.8) \]
where \( u_0(x) \in L_2(\omega) \) and \( f(x,t) \in C(L_2(\omega), I) \). Applying Theorem (3.1) to the mixed problem (3.6), (3.7) and (3.8), we deduce that this mixed problem has a unique solution

\[
u(x,t) \in W_2^{2m}(\omega)\]

and continuous with respect to \( t \in I \). Also this solution satisfies (3.4) and (3.5). Now from Sobolev’s embedding Theorem [7] we have the following corollary.

**Corollary 3.1.** If \( 2m > \frac{n}{2} + k \), then the solution of the mixed problem (3.6), (3.7) and (3.8)

\[
u(x,t) \in C^k(\omega),\]

and \( \frac{\partial u(x,t)}{\partial t} \) exists in the usual sense.

(2) Let \( X = L_p(R_n) \), \( 1 < p < \infty \), put \( G(t,x) = (2\sqrt{\pi t})^{-n} \exp(-|x|^2/4t) \), for each \( t > 0 \) and each \( x \in R_n \). The contraction semigroup \( \{T(t)\} \) defined by

\[
 T(t)u(x) = \int_{R_n} G(t,x-y)u(y)dy
\]

has the generator given by \( Au = -\Delta u \), with domain \( D(A) = W_p^2(R_n), 1 < p < \infty \), and \( A^* = A \). So this operator satisfies the assumptions of Theorem (2.2), therefore \( -A^{2m} = -\Delta^{2m} \) generates an analytic semigroup.

(3) Let the matrices \( a_j(t), j = 1, 2, \ldots, n \) and \( t > 0 \), be Hermitian. For each \( u = (u_1, u_2, \ldots, u_N)^t \in L_2(R_n)^N = X \), define

\[
 A(t)u = \sum_{j=1}^n a_j(t) \frac{\partial u}{\partial x_j} \tag{3.9}
\]

with domain \( D(A) = \{u \in X : A(t)u \in X\}CW_2^1(R_n)^N \). It is known [8] that for each \( \beta > 0 \) the operator \( \pm A(t) \in G(X,1,\beta) \) and \( A^* = -A \). So this operator (3.9) satisfies the assumptions of theorem (2.1), therefore \( (-1)^{m+1} A^{2m}(t) \) generates an analytic semigroup.

**References**


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