ON PRODUCTS OF CONJUGATE $EP_r$ MATRICES

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In this paper we answer the question of when product of conjugate $EP_r$ (con-$EP_r$) matrices is con-$EP_r$.

1. Introduction

Throughout this paper we deal with complex square matrices. Any matrix $A$ is said to be con-$EP$ if $R(A) = R(A^T)$ or equivalently $N(A) = N(A^T)$ or equivalently $AA^+ = A^+A$ and is said to be con-$EP_r$ if $A$ is con-$EP$ and $rk(A) = r$, where $R(A), N(A), A, A^T$ and $rk(A)$ denote the range space, null space, conjugate, transpose and rank of $A$ respectively [3]. $A^+$ denotes the Moore-Penrose inverse of $A$ satisfying the following four equations:

(1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, (4) $(XA)^* = XA$ [2].

$A^*$ is the conjugate transpose of $A$. In general product of two con-$EP_r$ matrices need not be con-$EP_r$. For instance, $\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$ are con-$EP_1$ matrices, but the product is not con-$EP_1$ matrix.

The purpose of this paper is to answer the question of when the product of con-$EP_r$ matrices is con-$EP_r$, analogous to that of $EP_r$ matrices studied by Baskett and Katz [1]. We shall make use of the following results on range space, rank and generalized inverse of a matrix.

(1) $R(A) = R(B) \Leftrightarrow AA^+ = BB^+$
(2) $R(A^+) = R(A^*)$
(3) $rk(A) = rk(A^+) = rk(A^T) = rk(A)$
(4) $(A^+)^+ = A$.

Results:

Theorem 1. Let $A_1$ and $A_n(n > 1)$ be con-EP$_r$ matrices and let $A = A_1A_2 \cdots A_n$. Then the following statements are equivalent.

(i) $A$ is con-EP$_r$.
(ii) $R(A_1) = R(A_n)$ and $rk(A) = r$
(iii) $R(A_1^*) = R(A_n^*)$ and $rk(A) = r$
(iv) $A^+$ is con-EP$_r$.

Proof. (i) $\iff$ (ii) : Since $R(A) \subseteq R(A_1)$ and $rk(A) = rk(A_1)$. We get $R(A) = R(A_1)$. Similarly, $R(A^T) = R(A^T_n)$. Now,

$A$ is con-EP$_r$ $\iff R(A) = R(A^T)$ and $rk(A) = r$

(by definition of con-EP$_r$)

$\iff R(A_1) = R(A^T_n) \ & \ rk(A) = r$

$\iff R(A_1) = R(A_n) \ & \ rk(A) = r$

( since $A_n$ is con-EP$_r$)

(ii) $\iff$ (iii) :

$R(A_1) = R(A_n) \iff A_1A_1^+ = A_nA_n^+$ (by result (1))

$\iff A_1A_1^+ = A_n^+A_n$

$\iff A_1^+A_1 = A_n^+A_n$ (since $A_1, A_n$ are con-EP$_r$)

$\iff R(A_1^+) = R(A_n^+)$ (by results (1) & (4))

$\iff R(A_1^*) = R(A_n^*)$ (by results (2)).

Therefore,

$R(A_1) = R(A_n)$ and $rk(A) = r \iff R(A_1^*) = R(A_n^*)$ and $rk(A) = r$.

(iv) $\iff$ (i) :

$A^+$ is con-EP$_r$ $\iff R(A^+) = R(A^+)^T$ and $rk(A^+) = r$
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Hence the Theorem.

**Corollary 1.** Let $A$ and $B$ be con-$E_P^r$ matrices. Then $AB$ is a con-$E_P^r$ matrix $\Leftrightarrow$ $rk(AB) = r$ and $R(A) = R(B)$.

*Proof.* Proof follows from Theorem 1 for the product of two matrices $A, B$.

Remark 1. In the above corollary both the conditions that $rk(AB) = r$ and $R(A) = R(B)$ are essential for a product of two con-$E_P^r$ matrices to be con-$E_P^r$. This can be seen in the following:

**Example 1.** Let $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}$ be con-$E_P^1$ matrices. Here $R(A) = R(B)$, $rk(AB) \neq 1$ and $AB$ is not con-$E_P^1$.

**Example 2.** Let $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} i & -i \\ i & i \end{bmatrix}$ be con-$E_P^1$ matrices. Here $R(A) \neq R(B)$, $rk(AB) = 1$ and $AB$ is not con-$E_P^1$.

Remark 2. In particular for $A = B$, Corollary 1 reduces to the following.

**Corollary 2.** Let $A$ be con-$E_P^r$. Then $A^k$ is con-$E_P^r$ $\Leftrightarrow$ $rk(A^k) = r$.

**Theorem 2.** Let $rk(AB) = rk(B) = r_1$ and $rk(BA) = rk(A) = r_2$. If $AB, B$ are con-$E_P^{r_1}$ and $A$ is con-$E_P^{r_2}$, then $BA$ is con-$E_P^{r_2}$.

*Proof.* Since $rk(BA) = rk(A) = r_2$, it is enough to show that $N(BA) = N(BA)^T$. $N(A) \subseteq N(BA)$ and $rk(BA) = rk(A)$ implies $N(BA) = N(A)$. Similarly, $N(AB) = N(B)$. Now,

$$N(BA) = N(A) = N(A^T) \quad (\text{Since } A \text{ is con-$E_P^{r_2}$})$$

$$\subseteq N(B^T A^T) = N((AB)^T) = N(AB) \quad (\text{Since } AB \text{ is con-$E_P^{r_1}$})$$

$$= N(B) \quad (\text{Since } N(AB) = N(B))$$
\[ N(B^T) \]  
\[
= N(B^T) \quad \text{(Since } B \text{ is con-EP,)} 
\subseteq N(A^T B^T) = N((BA)^T). 
\]

Further, \( rk(BA) = rk(BA)^T \) implies \( N(BA) = N(BA)^T \). Hence the 
Theorem.

**Lemma 1.** If \( A, B \) are con-EP matrices and \( AB \) has rank \( r \), then \( BA \) 
has rank \( r \).

**Proof.** \( rk(AB) = rk(B) - \dim(N(A) \cap N(B^*)^\perp) \). Since \( rk(AB) = 
rk(B) = r \), \( N(A) \cap N(B^*)^\perp = 0 \).

\[
N(A) \cap N(B^*)^\perp = 0 \Rightarrow N(A) \cap N(\bar{B})^\perp = 0 
\]  
\[
\text{(Since } B \text{ is con-EP,)} 
\Rightarrow N(\bar{A})^\perp \cap N(B) = 0 
\Rightarrow N(A^*)^\perp \cap N(B) = 0 
\]  
\[
\text{(Since } A \text{ is con-EP,)} 
\]

Now,

\[
rk(BA) = rk(A) - \dim(N(B) \cap N(A^*)^\perp) = r - 0 = r. 
\]

Hence the Lemma.

**Theorem 3.** If \( A, B \) and \( AB \) are con-EP matrices, then \( BA \) is con-EP.

**Proof.** Since \( A, B \) are con-EP matrices and \( rk(AB) = r \), by Lemma 1, 
\( rk(BA) = r \). Now the result follows from Theorem 2, for \( r_1 = r_2 = r \).

**Remark 3.** For any two con-EP matrices \( A \) and \( B \), since \( AB, \bar{AB}, A^+ B, 
AB^+, A^+ B^+, B^+ A^+ \) all have the same rank, the property of a matrix 
being con-EP is preserved for its conjugate and Moore-Penrose inverse, by 
applying Corollary 1 for a pair of con-EP matrices among \( A, B, A^+, B^+, 
\bar{A}, \bar{B}, A^+, \bar{B}^+ \) and using the result 2, we can deduce the following.

**Corollary 3.** Let \( A, B \) be con-EP matrices. Then the following statements 
are equivalent.

\begin{enumerate}
  
  \item \( AB \) is con-EP.
  \item \( \bar{AB} \) is con-EP.
  \item \( \bar{A^+ B} \) is con-EP.
  \item \( AB^+ \) is con-EP.
  \item \( A^+ B^+ \) is con-EP.
\end{enumerate}
(vi) $B^+A^+$ is con-$EP_r$.

**Theorem 4.** If $A, B$ are con-$EP_r$ matrices, $R(\bar{A}) = R(B)$ then $(AB)^+ = B^+A^+$.

**Proof.** Since $A$ is con-$EP_r$ and $R(\bar{A}) = R(B)$, we have $R(A^+) = R(B)$.

That is, given $x \in C_n$ (the set of all $n \times 1$ complex matrices) there exists a $y \in C_n$ such that $Bx = A^+y$. Now,

$$Bx = A^+y \Rightarrow B^+A^+ABx = B^+A^+AA^+y = B^+A^+y = B^+Bx.$$  

Since $B^+B$ is hermitian, it follows that $B^+A^+AB$ is hermitian. Similarly, $R(A^+) = R(B)$ implies $ABB^+A^+$ is hermitian.

Further by result (1), $A^+A = BB^+$. Hence,

$$AB(B^+A^+)AB = ABB^+(BB^+)B = AB$$

$$(B^+A^+)AB(B^+A^+) = B^+(BB^+)BB^+A^+ = B^+A^+.$$  

Thus $B^+A^+$ satisfies the defining equations of the Moore-Penrose inverse, that is, $(AB)^+ = B^+A^+$. Hence the Theorem.

**Remark 4.** In the above Theorem, the condition that $R(\bar{A}) = R(B)$ is essential.

**Example 3.** Let $A = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$ and $B = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$. Here $A$ and $B$ are con-$EP_1$ matrices, $rk(AB) = 1$, $R(\bar{A}) \neq R(B)$ and $(AB)^+ \neq B^+A^+$.

**Remark 5.** The converse of Theorem 4, need not be true in general. For,

Let $A = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$. $A$ and $B$ are con-$EP_1$ matrices, such that $(AB)^+ = B^+A^+$, but $R(\bar{A}) \neq R(B)$.

Next to establish the validity of the converse of the Theorem 4, under certain condition, first let us prove a Lemma.

**Lemma 2.** Let $A = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$ be an $n \times n$ con-$EP_r$ matrix where $E$ is an $r \times r$ matrix and if $[EF]$ has rank $r$, then $E$ is nonsingular. Moreover there is an $(n - r) \times r$ matrix $K$ such that $A = \begin{bmatrix} E & EK^T \\ KE & KEK^T \end{bmatrix}$. 

Proof. Since $A$ is con-EP$_r$, \[
\begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}
\] is con-EP$_r$ and $[E F]$ has rank $r$, the product \[
\begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
E & F \\
G & H
\end{bmatrix}
= \begin{bmatrix}
E & F \\
0 & 0
\end{bmatrix}
\] is a product of con-EP$_r$ matrices which has rank $r$. Therefore by Lemma 1 the product \[
\begin{bmatrix}
E & F \\
G & H
\end{bmatrix}
\begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
E & 0 \\
G & 0
\end{bmatrix}
\] has rank $r$. Hence there is an $(n-r) \times r$ matrix $K$ and an $r \times (n-r)$ matrix $L$ such that $G = KE$, $F = EL$, and $E$ is nonsingular.

Therefore,
\[
A = \begin{bmatrix}
E & EL \\
KE & KEL
\end{bmatrix}.
\]

Now, set $C = \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix}$ and consider
\[
CAC^T = \begin{bmatrix} I_r & 0 \\ -K & I_{n-r} \end{bmatrix} \begin{bmatrix} E & EL \\ KE & KEL \end{bmatrix} \begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix}
= \begin{bmatrix} E & EL \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & -K^T \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} E & -EK^T + EL \\ 0 & 0 \end{bmatrix}
\]
\[
CAC^T
\]
is con-EP$_r$. From $N(A) = N(CAC^T)$ it follows that $EL - EK^T = 0$, and so $L = K^T$, completing the proof.

**Theorem 5.** If $A, B$ are con-EP$_r$ matrices, $rk(AB) = r$ and $(AB)^+ = B^+ A^+$, then $R(\bar{A}) = R(\bar{B})$.

Proof. Since $A$ is con-EP$_r$, by Theorem 3 in [3], there is a unitary matrix $U$ such that, $U^T A U = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where $D$ is $r \times r$ nonsingular matrix.

Set $U^* B \bar{U} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$.

\[
U^T A B \bar{U} = U^T A U U^* B \bar{U}
= \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}
= \begin{bmatrix} DB_1 & DB_2 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} D & 0 \\ 0 & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}
\]
has rank $r$ and thus,

\[
U^* B A U = U^* \bar{B} \bar{U} U^T A U
= \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}
= \begin{bmatrix} B_1 D & 0 \\ B_3 D & 0 \end{bmatrix}
\]
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\[
\begin{bmatrix}
B_1 & 0 \\
B_3 & 0
\end{bmatrix}
\begin{bmatrix}
D & 0 \\
0 & I_{n-r}
\end{bmatrix}
\]

has rank \( r \). It follows that \[
\begin{bmatrix}
B_1 & B_2 \\
0 & 0
\end{bmatrix}
\]
and \[
\begin{bmatrix}
B_1 & 0 \\
B_3 & 0
\end{bmatrix}
\]
have rank \( r \), so that \( B_1 \) is nonsingular.

By Lemma 2, \( U^*B\bar{U} = \begin{bmatrix}
B_1 & B_1K^T \\
KB_1 & KB_1K^T
\end{bmatrix}, \) with \( rk(U^*B\bar{U}) = rk(B_1) = r \). By using Penrose representation for the generalized inverse [4], we get

\[
(U^*B\bar{U})^+ = \begin{bmatrix}
B_1^*PB_1^* & B_1^*PB_1^*K^* \\
\bar{K}B_1^*PB_1^* & \bar{K}B_1^*PB_1^*K^*
\end{bmatrix}
\]

where \( P = (B_1B_1^* + B_1K^T\bar{K}B_1^*)^{-1}B_1(B_1^*B_1 + B_1^*K^*KB_1)^{-1} \)

\[
U^TB^+U = (U^*B\bar{U})^+ = \begin{bmatrix}
Q & QK^* \\
\bar{K}Q & \bar{K}QK^*
\end{bmatrix}
\]

where \( Q = (I + K^T\bar{K})^{-1}B_1^{-1}(I + K^*K)^{-1} \)

\[
U^*A^+\bar{U} = (U^TAU)^+ = \begin{bmatrix}
D^{-1} & 0 \\
0 & 0
\end{bmatrix}
\]

\[
U^TAB\bar{U} = U^TAB\bar{U}(U^TAB\bar{U})^+U^TAB\bar{U}
\]

\[
= U^TAB\bar{U}(U^T(AB)^+\bar{U})U^TAB\bar{U} \quad \text{(since \( U \) is unitary)}
\]

\[
= U^TAB\bar{U}(U^TB^+A^+\bar{U})U^TAB\bar{U} \quad \text{(by hypothesis)}
\]

\[
= U^TAB\bar{U}(U^TB^+U)(U^*A^+\bar{U})U^TAB\bar{U} \quad \text{(since \( U \) is unitary)}
\]

On simplification, we get,

\[
DB_1QB_1 + DB_2\bar{K}QB_1 = DB_1
\]

\[
\Rightarrow DB_1(I + B_1^{-1}B_2\bar{K})QB_1 = DB_1.
\]

Since \( B_2 = B_1K^T \), \( QB_1 = (I + K^T\bar{K})^{-1} \). Hence \( (I + K^T\bar{K}) = (QB_1)^{-1} = I \). Thus \( K^T\bar{K} = 0 \) which implies \( K^*K = 0 \) so that \( K = 0 \).

\[
U^*B\bar{U} = \begin{bmatrix}
B_1 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
U^TAU = \begin{bmatrix}
D & 0 \\
0 & 0
\end{bmatrix} \Rightarrow U^*\bar{A}\bar{U} = \begin{bmatrix}
\bar{D} & 0 \\
0 & 0
\end{bmatrix}.
\]
Since $\bar{D}$ and $B_1$ are $r \times r$ nonsingular matrices we have

$$R(\bar{D}) = R(B_1) \Rightarrow R\left( \begin{bmatrix} \bar{D} & 0 \\ 0 & 0 \end{bmatrix} \right) = R\left( \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\Rightarrow R(U^*\bar{A}U) = R(U^*B\bar{U})$$

$$\Rightarrow R(\bar{A}) = R(B).$$

Hence the Theorem.

**Theorem 6.** Let $A, B$ are con-EP$_{r}$ matrices, $rk(AB) = r$ and $(AB)^+ = A^+B^+$, then $AB$ is con-EP$_{r}$.

**Proof.**

$$R(B) = R(B^T) \quad \text{(since $B$ is con-EP$_{r}$)}$$

$$\Rightarrow R(\bar{B}) = R(B^*)$$

$$= R(B^*A^*) \quad \text{(since $R(B^*A^*) \subseteq R(B^*)$ and $rk(AB)^* = rk(AB) = r = rk(B^*)$)}$$

$$= R(AB)^* = R(AB)^+ \quad \text{(by result (2))}$$

$$= R(A^+B^+) \quad \text{(by hypothesis)}$$

$$\subseteq R(A^+) = R(A^*) = R(\bar{A})$$

(by result (2) & $A$ is con-EP$_{r}$).

$$\Rightarrow R(\bar{B}) = R(\bar{A}) \Rightarrow R(B) = R(A).$$

Since $rk(AB) = r$ and $R(B) = R(A)$, by Corollary 1, $AB$ is con-EP$_{r}$.

Hence the Theorem.

**References**


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