ON *-PRIMES AND *-VALUATIONS

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We study the relations between *-primes and *-valuations to deduce a necessary and sufficient condition for extending *-valuations.

1. Introduction

Let \((D, *)\) be a *-field; that is, a skew field with an involution * (an anti-automorphism of order 2). For general valuation theory on skew fields one can refer to [7]. For *-fields we need our valuations to also be compatible with the involution *. Following Holland [4], we define a *-valuation on a *-field \((D, *)\) to be a valuation \(w\) onto an additively written ordered group with the additional property that \(w(x^*) = w(x)\) for all non-zero \(x \in D\). One of the properties any reasonable generalization of the concept of a valuation should have is that valuations allow extensions to larger fields. In [3] and [8] a necessary and sufficient condition is given for extending an abelian valuation from a division ring \(D\) to the over division ring \(E\). We will solve here a *-version of this problem were valuations are replaced with *-valuations. This is done by using a characterization of those *-primes giving rise to *-valuations, together with an extension theorem for *-primes. Basic properties of *-primes in a *-ring \(R\) are given in section (2).

2. *-primes

Throughout this section \(R\) will be an arbitrary *-ring with unit. A couple \((P, R')\) is said to be a *-prime in the *-ring \(R\) if the following conditions are satisfied:
1) \(R'\) is a *-closed subring of \(R\).
2) \(P\) is a *-closed prime ideal in \(R'\).

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3) if $xR'y \subset P$ with $x, y \in R$ then $x \in P$ or $y \in P$. 
If $P$ is a $\ast$-closed prime ideal of $R$, then $(P, R)$ is a $\ast$-prime in $R$.

Let $T$ be a subset of the $\ast$-ring $R$. A subset $S$ of $R$ is an $m$-system for $T$ iff $0 \notin S$ and for any $s_1, s_2 \in S$ there is an $x \in T$ such that $s_1xs_2 \in S$.

**Lemma 1.** If $(P, R')$ is a $\ast$-prime in $R$ then $R - P$ is an $m$-system for $R'$. Conversely, if $P$ is a $\ast$-closed additive subgroup of $R'$ which is multiplicatively closed and such that $R - P$ is an $m$-system for $R' = \{ r \in R | rP \subset P \text{ and } Pr \subset P \}$ then $(P, R')$ is a $\ast$-prime in $R$.

**Proof.** This is immediate.

Consider $S = \{(P, R') | R' \text{ a } \ast\text{-closed subring of } R, P \text{ a } \ast\text{-closed prime ideal of } R'\}$. Let $(P_1, R_1)$ and $(P_2, R_2)$ be elements of $S$. Say that $(P_1, R_1)$ dominates $(P_2, R_2)$ (notation: $(P_2, R_2) < (P_1, R_1)$) iff $R_1 \supset R_2$ and $P_1 \cap R_2 = P_2$. If $(P, R')$ is maximal in $S$ with respect to $<$ then call it a dominating pair in $R$.

**Lemma 2.** Let $(P, R^p)$ be a dominating pair in $R$. If $R'$ is a $\ast$-closed subring of $R$, $I$ a $\ast$-closed ideal in $R'$ such that $R^p \subset R'$ and $I \cap R^p = P$ then $I = P$ and $R' = R^p$.

**Proof.** Let $T = \{(I, R') | I \text{ a } \ast\text{-closed ideal in } R^p, R^p \subset R' \text{ and } I \cap R' = P\}$. Since $T$ is not empty it contains (by Zorn's lemma) a maximal element, say $(Q, B)$. One can prove that $Q$ is a $\ast$-closed prime ideal in $B$. This yields that $(Q, B) > (P, R^p)$. But $(P, R^p)$ is a dominating pair, hence $P = Q$, $B = R^p$ follows, i.e. $T = \{(P, R^p)\}$ which proves the lemma.

By a $\ast$-$R$-ring $A$ we mean a $\ast$-ring $A$ where $R$ is assumed to be a subring of $A$.

**Lemma 3.** Let $\pi = (P, R')$ be an arbitrary $\ast$-prime in $R$ then $(K, A^k)$ is a $\ast$-prime in $A$ which restricts to $\pi$ (i.e. $K \cap R = P$ and $A^k \cap R = R'$) if and only if

1) $K$ is a $\ast$-closed left and right $R'$-module.
2) $K \cap R = P$.
3) $A - K$ is an $m$-system for $A^k$.

**Proof.** If $(K, A^k)$ is a $\ast$-prime in $A$ which restricts to $\pi$, then (1) and (2) are evident, and (3) holds by using Lemma (1). The converse is also true by using Lemma (1).

If $S, T$ are subsets of $A$ then $S < T >$ stands for $\{ x \in A | x = \sum s_i t_i, s_i \in S, t_i \in T \}$. 
Theorem 4. Let \( A \) a \(*\)-algebra, \( \pi = (P, R^p) \) a fixed dominating \(*\)-prime in \( R \). Let \( B \) be a \(*\)-closed subset of \( A \) and \( M \) a \(*\)-closed subset of \( B \) satisfy the following properties:

(i) \( BP \subseteq P \prec B \succ \) and \( BR^p \subseteq R^p \prec B \succ \).
(ii) \( P \prec B \succ \cap R = P \).
(iii) \( M \) is an \( m \)-system for \( B \).
(iv) \( R^p - P \subseteq M \).
(v) \( M \cap P \prec B \succ = \emptyset \).

Then there is a \(*\)-prime \( (K, A') \) in \( A \) which restricts to \( \pi \), such that \( BK \subseteq K \) and \( KB \subseteq K \).

One can adapt the proof of Theorem (2.3) in [8] to prove Theorem (4).

3. \(*\)-valuations in \(*\)-fields

Let \( (D, *) \) be a \(*\)-field, and let \( D^x \) be the multiplicative group of non-zero elements of \( D \). Following Holland [4], a function \( w \) from \( D^x \) onto an additively written ordered group \( \Gamma \) is called a \(*\)-valuation of \( D \) if

(i) \( w(xy) = w(x) + w(y) \), for every \( x, y \in D^x \).
(ii) \( w(x + y) \geq \min(w(x), w(y)), x + y \neq 0 \).
(iii) \( w(x^*) = w(x) \).

It then follows that \( \Gamma \) is abelian since \( w(x) + w(y) = w(xy) = w(y^*x^*) = w(y) + w(x) \).

First, we recall some basic facts about \(*\)-valuations in \(*\)-fields.

Definition. Let \( R \) be a subring of \( D \).

(1) \( R \) is called total if for every \( x \in D^x \), \( x \) or \( x^{-1} \in R \).
(2) \( R \) is called symmetric if it contains \( x^*x^{-1} \) for every \( x \in D^x \).
(3) \( R \) is called \(*\)-valuation ring if it is total and symmetric.

Remark 5 [4]. If \( R \) is a symmetric subring of \( D \) then \( R \) is \(*\)-closed and preserved under conjugation.

Remark 6. If \( R \) is a \(*\)-closed total subring which is preserved under conjugation then \( R \) is symmetric and so it is a \(*\)-valuation subring.

Lemma 7[4]. Let \( w \) be a \(*\)-valuation of a \(*\)-field \( D \), then

(1) \( V = \{ x \in D \mid w(x) \geq 0 \} \) is \(*\)-closed subring of \( D \) and \( P = \{ x \in D \mid w(x) > 0 \} \) is a \(*\)-closed maximal ideal of \( V \).
(2) $V$ is total.
(3) Every ideal in $V$ is two-sided.
(4) The ideal $P$ is the unique maximal ideal of $V$, formed by the non-units in $V$, and $V/P$ is a $*$-skew-field.
(5) $V$ is symmetric (therefore preserved under conjugation).

**Proposition 8 [4].** Given a $*$-valuation subring $V$ of the $*$-field $D$, then there exist an ordered abelian group $\Gamma$ and a $*$-valuation $\nu: D^x \to \Gamma$ such that $V$ coincides with the $*$-valuation ring of $W$.

In a $*$-field which is finite-dimensional over its centre, $*$-valuation rings may be defined without demanding symmetry.

**Theorem 9.** Let $D$ be a $*$-field finite dimensional over its centre. Then any $*$-closed total subring $V$ of $D$ is a $*$-valuation subring.

One can adopt the proof of Theorem (3) in [2] to prove theorem (9).

The following proposition characterises those $*$-primes in $*$-fields which yield $*$-valuation rings.

**Proposition 10.** A $*$-prime $(P, D^p)$ such that $D^p$ is preserved under conjugation, yields a $*$-valuation of $D$ with $*$-valuation ring $D^p$ and maximal ideal $P$. Conversely, if $V$ is a $*$-valuation ring, $P$ its maximal ideal, then $(P, V)$ is a $*$-prime of $D$.

**Proof.** We first claim that $D^p$ is total. Suppose $x^{-1} \notin D^p$ then, since $xD^px^{-1} \subset D^p$, $PxD^px^{-1} \subset P$ follows, but this yields $Px \subset P$ (using the defining property of $*$-primes). On the other hand also $xP \subset P$. Hence $x \in D^p$ and $D^p$ is total. Clearly $D^p$ is $*$-closed. Then, by Remark (6), $D^p$ is a $*$-valuation ring. Now, if $x \notin P$ then as before one can show that $x^{-1} \in D^p$, a contradiction, so $x \in P$. This proves that $P$ is maximal ideal in $D^p$.

For the converse, it is enough to check property (3) in the definition of $*$-primes: take $x, y \in D$ such that $xy \in P$, if $x \notin P$ then $x^{-1} \in V$ and so $x^{-1}xy = y \in P$.

**Theorem 11.** Let $D, E$ be $*$-fields, $D \subset E$, a $*$-valuation $\nu$ on $D$ extends to a $*$-valuation of $E$ if and only if $PE^c$ is a proper ideal in $VE^c$, where $V$ is the $*$-valuation subring of $w$, $P$ its maximal ideal, and $E^c$ is the commutator subgroup of $E$.

**Proof.** We first note that $PE^c = E^cP$ and $VE^c = E^cV$ (for, $r(xy^{-1}y^{-1}) = ((rxy)r(rxy)^{-1}r^{-1})r$). Let $B = VE^c$ and $M = V - P$. Clearly $M \subset B \subset$
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Since both \(V\) and \(E^c\) are \(*\)-closed, it follows that \((VE^c)^* = (E^c)^*V^* = E^cV = VE^c\), that is \(B\) is \(*\)-closed. Also, from the fact that \(w(x^*) = w(x)\), it follows that \(M\) is \(*\)-closed. By Proposition (10), \((P,V)\) is a \(*\)-prime in \(D\). Applying Theorem (4), with \(B = VE^c\) and \(M = V - P\) which satisfy properties (i)-(v), yields a \(*\)-prime \((P', V')\) in \(E\) such that \(VE^c \subseteq E^{p'}\) and \(P' \cap D = P\). Clearly \((P', E^{p'})\) is a \(*\)-prime in \(E\).

To show that \(E^{p'}\) is a \(*\)-valuation ring in \(E\) with maximal ideal \(P'\), one must show that \(E^{p'}\) is preserved under conjugation (Proposition (10)). Let \(x \in E\) and \(e \in E^{p'}\), then \(xe^{-1}e^{-1} \in E^c \subseteq E^{p'}\), so that \(xe^{-1} \in E^{p'}\). Now, by Proposition (8), \(E^{p'}\) defines the desired extension.

To prove the converse, assume that \(W_1\) is an extension of \(w\) and \(PE^c\) is not a proper ideal in \(VE^c\). Then an equation of the form

\[
\sum_i a_i c_i = 1, \quad a_i \in P, \quad c_i \in E^c
\]

holds. Thus \(0 = w_1(1) \geq \min_i \{w(a_i) + w_1(c_i)\}\). Since \(a_i \in P, w(a_i) > 0\) follows. Also \(w_1(c_i) = 0\) (for, \(c_i\) is a product of commutators), so \(w(a_i) + w_1(c_i) > 0\) a contradiction. Therefore \(PE^c\) is a proper ideal in \(VE^c\).

For examples of a \(*\)-field extension where the condition in Theorem (11) does not hold, see [8].

There is a remarkable relation between \(*\)-valuations and the notion of ordering of a \(*\)-field \(D\). Beginning with a definition of Baer, at least four different notions of orderings have been proposed for \(D\) [1, 4, 5, 6]. The connection between orderings and \(*\)-valuation is provided by the fact that, to any ordering \(\geq\) on \(D\), we can associate a \(*\)-valuation ring \(V\) (the order subring) consists of elements of \(D\) which are bounded by some rational numbers with respect to \(\geq\). This has been done for the \(c\)-ordering [1], the strong ordering [5], and the Jorden ordering [6]. For the notion of Baer ordering [4], it is shown that \(V\) is a total subring, but whether or not \(V\) is a \(*\)-valuation ring is still on open question. The following corollary of Theorem (9) is a partial answer to that question.

**Corollary.** If \(D\) is a Baer ordered \(*\)-field which is finite dimensional over its centre, then the order subring \(V\) associated with the ordering is a \(*\)-valuation ring.
References


