SASAKIAN MANIFOLDS WITH VANISHING CONTACT BOCHNER CURVATURE TENSOR

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1. Introduction

Let $M$ be a $(2m+1)$-dimensional differentiable manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, where, hear and in the sequel, the indices $h, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2m + 1\}$ and let $M$ admit an almost contact structure, that is, a set $(\phi^j_i, \xi^j, \eta_i)$ of a tensor field $\phi^j_i$ of type $(1,1)$, a vector field $\xi^j$ and 1-form $\eta_i$ satisfying

\[
\phi^j_i \phi^i_j = -\delta^j_i + \eta_i \xi^j, \quad \eta_i \phi^i_i = 0, \quad \phi^j_i \xi^i = 0, \quad \eta_i \xi^i = 1.
\]

We now assume that $M$ admit an almost contact metric structure, that is, a set $(\phi^j_i, \xi^j, \eta_i, g_{ji})$ of $\phi^j_i, \xi^j, \eta_i$ and positive definite Riemannian metric $g_{ji}$ satisfying, in addition to (1.1),

\[
g_{st} \phi^s_j \phi^t_i = g_{ji} - \eta_j \eta_i, \quad \eta_j = g_{ji} \xi^i, \quad g_{ji} \xi^j \xi^i = 1.
\]

In this case, we call $M$ an almost contact metric manifold. Comparing the first equation of (1.1) and (1.2), we see that $\phi_{ji} = \phi^j_i g_{li}$ is skew-symmetric. Since, in an almost contact metric manifold, we have the first equation of (1.3), we shall write $\eta^h$ instead of $\xi^h$ in the sequel. We denote by $\nabla_i, K_{kji}^h, K_{ji}$ and $K$ the operator of covariant differentiation with respect to the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature respectively.

Received September 1, 1991.
2. Sasakian manifold

In this section, we consider a Sasakian manifold $M$. In a Sasakian manifold $M$, we have

\begin{align*}
\nabla_k \phi_j^i &= -g_{kj} \eta^i + \delta_i^j \eta_j, \\
\nabla_j \eta^i &= \phi_j^i. \\
\end{align*}

Now from (2.1), (2.2) and the Ricci identity

\begin{align*}
\nabla_k \nabla_j \eta^h - \nabla_j \nabla_k \eta^h &= K_{kjt} \eta^t, \\
\end{align*}

we find

\begin{align*}
K_{kjt} \eta^t &= \delta^h_k \eta_j - \delta^h_j \eta_k \\
\end{align*}

or

\begin{align*}
K_{kji} \eta_t &= \eta_k g_{ji} - \eta_j g_{ki}, \\
\end{align*}

from which, by contraction,

\begin{align*}
K_j^t \eta_t &= 2m \eta_j. \\
\end{align*}

From equation (2.1), (2.2) and the Ricci identity

\begin{align*}
\nabla_k \nabla_j \phi_i^h - \nabla_j \nabla_k \phi_i^h &= K_{kjt} \phi_i^t - K_{kji} \phi_j^t, \\
\end{align*}

we find

\begin{align*}
K_{kjt} \phi_i^t - K_{kji} \phi_j^t &= -\phi_k^h g_{ji} + \phi_j^h g_{ki} - \delta_k^h \phi_{ji} + \delta_j^h \phi_{ki}, \\
\end{align*}

from which, by contraction,

\begin{align*}
K_{ji} \phi_i^t + K_{tij} \phi^t_s &= -(2m - 1) \phi_j. \\
\end{align*}

Since

\begin{align*}
K_{tij} \phi^t_s &= K_{aijt} \phi^t_s = -K_{tij} \phi^t_s, \\
\end{align*}

we have from (2.8)

\begin{align*}
K_{ji} \phi_i^t + K_{ij} \phi_j^t &= 0. \\
\end{align*}

Since

\begin{align*}
K_{tij} \phi^t_s &= \frac{1}{2} (K_{tij} - K_{sjit}) \phi^t_s = -\frac{1}{2} K_{tij} \phi^t_s
\end{align*}
we have from (2.8)

\[(2.10) \quad K_{\tau\delta ji} i^t s = 2K_{j l} i^t + 2(2m - 1)\phi_{ji}.\]

Transvecting (2.7) with \(\phi_i^l\) and using (2.5), we find

\[-K_k j^l - K_{kji}^l \phi_i^t \phi_l^h = -\phi_k^h \phi_t j + \phi_j^h \phi_t k - \delta_k^h g_{jl} + \delta_j^h g_{kl};\]

or

\[(2.11) \quad K_{kji}^j \phi_i^l \phi_t^h = K_{kji}^j \phi_i^l + \phi_k^l \phi_j^h - \phi_j^h \phi_k^l - g_{kh} g_{ji} + g_{jh} g_{ki}.\]

Transvecting (2.11) with \(\phi_s^h \phi_m^j\), we find

\[(2.12) \quad K_{kjsl} \phi_m^j \phi_i^l - K_{kji}^l \phi_m^j \phi_s^h\]

\[= -g_{ks}(g_{mi} - \eta_m \eta_i) + g_{ki}(g_{ms} - \eta_m \eta_s) + \phi_{ks} \phi_{mi} - \phi_{ki} \phi_{ms},\]

or

\[(2.13) \quad K_{iktj}^l \phi_i^t \phi_s^m - K_{kltj}^l \phi_i^t \phi_s^m = \Lambda_{kji},\]

where we have put

\[(2.14) \quad \Lambda_{kji} = -g_{kj}(g_{mi} - \eta_m \eta_i) + g_{ki}(g_{mj} - \eta_m \eta_j) + \phi_{kj} \phi_{mi} - \phi_{ki} \phi_{mj}.\]

From (2.13), we have easily

\[(2.15) \quad K_{jkt s} \phi_i^t \phi_m^s - K_{ikts} \phi_j^t \phi_m^s = \Lambda_{kji}.\]

From

\[(2.16) \quad K_{kljt}^l \phi_i^t \phi_t^h - K_{kltj}^l \phi_i^t \phi_t^h = K_{kljt}^l (\phi_i^l \phi_t^h - \phi_t^h \phi_i^l)\]

and (2.11), we have

\[(2.17) \quad \Omega_{kji} = K_{kji} + \phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - g_{kh} g_{ji} + g_{jh} g_{ki}.\]

Moreover we have

\[(2.18) \quad K_{ksjt} \phi_i^t \phi_t^h = K_{jskt} \phi_s^h \phi_i^t.\]
3. Sasakian manifold with vanishing contact Bochner curvature tensor

We consider the section determined by \( \phi X \) and \( \phi^2 X \) which are orthogonal each other ([3]). We call this a \( C \)-holomorphic section, and the sectional curvature determined by such a section is said to be the \( C \)-holomorphic sectional curvature:

\[
K(X) = -\frac{K_{kjih}(\phi X)^k(\phi^2 X)^j(\phi X)^i(\phi^2 X)^h}{g_{kj}(\phi X)^k(\phi X)^j g_{ih}(\phi^2 X)^i(\phi^2 X)^h},
\]

from which, using (2.5), we have

\[
K(X) = -\frac{[K_{ksir} \phi_u^k \phi_i^r + \eta_u \eta_r] X^u X^s X^t X^r}{\gamma_{us} \gamma_{tr} X^u X^s X^t X^r},
\]

where we have put

\[\gamma_{kj} = g_{kj} - \eta_k \eta_j.\]

Now, suppose that the contact Bochner curvature tensor \( B_{kji}^h \) ([4]) of the Sasakian manifold \( M \) of dimension \( 2m + 1 > 3 \) ([2]) vanishes, then we have

\[
K_{kji} = -\gamma_{kh} L_{ji} + \gamma_{jh} L_{ki} - L_{kh} \gamma_{ji} + L_{jh} \gamma_{ki} - \phi_{kh} M_{ji} + \phi_{jh} M_{ki} - M_{kh} \phi_{ji} + M_{jh} \phi_{ki} + 2(M_{kj} \phi_{ih} + \phi_{kj} M_{ih}) - (\phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - 2 \phi_{kj} \phi_{ih}),
\]

where

\[
L_{ji} = -\frac{1}{2(m+2)} [K_{ji} + (L + 3) g_{ji} - (L - 1) \eta_j \eta_i], \quad L = g^{ji} L_{ji},
\]

\[M_{ji} = -L_{ji} \phi_i^t,
\]

and consequently the \( C \)-holomorphic sectional curvature \( K(X) \) with respect to a \( C \)-holomorphic section spanned by \( \phi X \) and \( \phi^2 X \) is given by

\[
K(X) = -\frac{1}{\gamma_{su} X^s X^u} 8(L_{tr} + \eta_t \eta_r) X^t X^r - 3,
\]

where we have used

\[
L_{st} \phi_j^s \phi_i^t = L_{ji} + \eta_j \eta_i,
\]

\[
L_{ji} \eta^i = -\eta_j.
\]
Conversely suppose that the sectional curvature $K(X)$ of $M$ defined by (3.1) is equal to the right-hand side member of (3.4). Then from (3.2) and (3.4), we have

$$
(K_{ksir} \phi_u^k \phi_t^i + \eta_{st} \gamma_{ut} - \{8 \gamma_{sr} (L_{ut} + \eta_{ut}) + 3 \gamma_{sr} \gamma_{ut}\} X^u X^s X^t X^r = 0,
$$
or

$$
K_{ktis} \phi_j^t \phi_h^s X^k X^j X^i X^h
$$

where $L_{kji} X^i X^j$ is a certain quadratic form whose coefficients satisfy (3.5).

This place $X$'s are arbitrary, therefore by (2.18) and the symmetry of $L_{jh}$, we have from (3.8), ([1])

$$
2(K_{ktis} \phi_j^t \phi_h^s + K_{kths} \phi_i^t \phi_j^s + K_{ktis} \phi_h^t \phi_j^s + K_{kths} \phi_i^t \phi_k^s + K_{ktjs} \phi_j^t \phi_i^s + K_{kths} \phi_j^t \phi_k^s + K_{jtis} \phi_k^t \phi_j^s + K_{jtis} \phi_i^t \phi_k^s)
$$

$$
= 4[\gamma_{ki} (8L_{jih} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih}) + \gamma_{kj} (8L_{kij} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih})]
$$

$$
+ \gamma_{kh} (8L_{kij} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih}) + \gamma_{ij} (8L_{kij} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih})
$$

$$
+ \gamma_{ih} (8L_{kij} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih}) + \gamma_{ij} (8L_{kij} + 7\gamma_{ki} \gamma_{ij} + 3\gamma_{ih})]
$$

from which, taking account of (2.13), (2.15) and (2.16), we have

$$
(3K_{ktis} \phi_j^t \phi_h^s + 2 \Lambda_{kijh} + 4 \Lambda_{ijhk} + 2\Omega_{jihk})
$$

$$
+(8K_{kths} \phi_i^t \phi_j^s + 4 \Lambda_{hijk} + 2 \Lambda_{kijh} + 2\Omega_{hijk})
$$

$$
+(8K_{ktjs} \phi_j^t \phi_i^s + 2 \Lambda_{khij} + 4 \Lambda_{ijhk} + 2\Omega_{hijk})
$$

$$
= 4[8(\gamma_{ki} L_{jih} + \gamma_{kj} L_{kij} + \gamma_{kh} L_{jih} + \gamma_{ij} L_{kij} + \gamma_{ih} L_{kij} + \gamma_{ij} L_{kij})
$$

$$
+ 7(\gamma_{ki} \eta_{ij} + \gamma_{kj} \eta_{ij} + \gamma_{kh} \eta_{ij} + \gamma_{ij} \eta_{ki} + \gamma_{ih} \eta_{kj} + \gamma_{ij} \eta_{kh})
$$

$$
+ 6(\gamma_{ki} \gamma_{ij} + \gamma_{kj} \gamma_{ij} + \gamma_{kh} \gamma_{ij})]
$$

Taking into consideration of the definition of the tensors $\Lambda$ and $\Omega$, we see that

$$
2(\Lambda_{kijh} + \Lambda_{kijh} + \Lambda_{kijh}) = 4(\phi_{kh} \phi_{ji} + \phi_{kj} \phi_{ih} + \phi_{ki} \phi_{hj}),
$$

$$
4(\Lambda_{ijhk} + \Lambda_{hijk} + \Lambda_{jihk}) = 4(g_{hi} \eta_{ij} \eta_{k} - g_{jk} \eta_{ij} \eta_{h} + g_{ij} \eta_{ki} \eta_{h}
$$

$$
- g_{hk} \eta_{ij} \eta_{i} + g_{jkh} \eta_{ij} \eta_{k} - g_{ikh} \eta_{ij} \eta_{j}) + 8(\phi_{hi} \phi_{jk} + \phi_{jki} \phi_{hk} + \phi_{jki} \phi_{hk})
$$
and

\[ (3.13) \quad 2(\Omega_{ijk} + \Omega_{hjk} + \Omega_{hji}) = 4(\phi_{kh} \phi_{ij} + \phi_{kj} \phi_{hi} + \phi_{ki} \phi_{jh}) , \]

where we have used

\[ K_{hkji} + K_{kjhi} + K_{jhki} = 0. \]

Substituting (3.11), (3.12) and (3.13) into (3.10), we find

\[
2(K_{kts} \phi_{j}^{t} \phi_{h}^{s} + K_{kts} \phi_{i}^{t} \phi_{j}^{s} + K_{kts} \phi_{h}^{t} \phi_{i}^{s}) \\
+ (g_{hi} \eta_{j} \eta_{k} - g_{jk} \eta_{i} \eta_{h} + g_{ji} \eta_{k} \eta_{h} - g_{hj} \eta_{k} \eta_{i} - g_{ik} \eta_{j} \eta_{h}) \\
+ 2(\phi_{ji} \phi_{k} + \phi_{jk} \phi_{i} + \phi_{ki} \phi_{h}) \\
= 8(\gamma_{i} L_{jh} + \gamma_{jk} L_{hi} + \gamma_{kh} L_{ji} + \gamma_{ij} L_{hk} + \gamma_{ih} L_{kj} + \gamma_{jh} L_{ki}) \\
+ 7(\gamma_{ki} \eta_{j} \eta_{h} + \gamma_{kj} \eta_{i} \eta_{h} + \gamma_{kh} \eta_{j} \eta_{i} + \gamma_{ij} \eta_{k} \eta_{h} + \gamma_{ih} \eta_{k} \eta_{j} + \gamma_{jh} \eta_{i} \eta_{h}) \\
+ 6(\gamma_{ki} \gamma_{jh} + \gamma_{kj} \gamma_{ih} + \gamma_{kh} \gamma_{ji}),
\]

or

\[
(3.14) \quad 2(K_{kts} \phi_{q}^{t} \phi_{p}^{s} + K_{kps} \phi_{i}^{t} \phi_{q}^{s} + K_{kqs} \phi_{p}^{t} \phi_{i}^{s}) \\
+ (g_{pi} \eta_{q} \eta_{k} - g_{qk} \eta_{i} \eta_{p} + g_{qi} \eta_{k} \eta_{p} - g_{pk} \eta_{i} \eta_{q} + g_{pq} \eta_{i} \eta_{k} - g_{ik} \eta_{q} \eta_{p}) \\
+ 2(\phi_{pi} \phi_{q} + \phi_{pq} \phi_{i} + \phi_{qi} \phi_{p}) \\
= 8(\gamma_{i} L_{qp} + \gamma_{jq} L_{ip} + \gamma_{kp} L_{qi} + \gamma_{iq} L_{pk} + \gamma_{ip} L_{kp} + \gamma_{qp} L_{ki}) \\
+ 7(\gamma_{ki} \eta_{q} \eta_{p} + \gamma_{kq} \eta_{i} \eta_{p} + \gamma_{kp} \eta_{q} \eta_{i} + \gamma_{iq} \eta_{p} \eta_{k} + \gamma_{ip} \eta_{q} \eta_{k} + \gamma_{qp} \eta_{i} \eta_{k}) \\
+ 6(\gamma_{ki} \gamma_{qp} + \gamma_{kq} \gamma_{ip} + \gamma_{kp} \gamma_{qi}),
\]

from which, transvecting with \( \phi_{j}^{q} \phi_{h}^{p} \), we have

\[
(3.15) \quad 2(K_{kji} - K_{ktp} \phi_{i}^{t} \phi_{h}^{p} - K_{kq} \phi_{j}^{q} \phi_{i}^{s}) \\
- g_{hk} \eta_{j} \eta_{i} + g_{ki} \eta_{j} \eta_{h} - g_{ji} \eta_{k} \eta_{h} + g_{ih} \eta_{k} \eta_{j} \\
+ \gamma_{jh} \eta_{i} \eta_{h} + 2(\gamma_{ij} \eta_{h} - \phi_{ij} \phi_{h} - \gamma_{kj} \gamma_{ij}) \\
= 8[\gamma_{ki}(L_{jh} + \eta_{j} \eta_{h}) - \phi_{jk} M_{ih} - \phi_{hk} M_{ij} \\
- \phi_{ji} M_{kh} - \phi_{hi} M_{kj} + \gamma_{jh} L_{ki}] \\
+ 7\gamma_{jh} \eta_{k} \eta_{i} + 6(\gamma_{ki} \gamma_{jh} + \phi_{jk} \phi_{hi} + \phi_{ji} \phi_{hk}),
\]

where we have used (2.5) and (3.5) and have put

\[
(3.16) \quad M_{ji} = -L_{jt} \phi_{j}^{t},
\]
or

\[ K_{kjih} - K_{ktpj} \phi_i^t \phi_h^p - K_{khji} + \phi_{hi} \phi_{kj} + g_{ki} g_{hj} - g_{kh} g_{ij} + g_{ki} \eta_h - g_{kj} \eta_i \]

\[ = 4[\gamma_{ki}(L_{jh} + \gamma_{ji}) - \phi_{jk} M_{ih} - \phi_{kk} M_{ij} - \phi_{ji} M_{kh} - \phi_{ki} M_{kj} + \gamma_{jh} L_{ki}] \]

\[ + 3(\gamma_{jh} \eta_i \gamma_{ki} + \gamma_{ki} \gamma_{jh} + \phi_{jk} \phi_{hi} + \phi_{ji} \phi_{hk}), \]

where by (2.11), we have used

\[ K_{khqs} \phi_j^q \phi_i^s = K_{khji} + \phi_{ki} \phi_{hj} - \phi_{hi} \phi_{kj} - g_{ki} g_{hj} + g_{hi} g_{kj}. \]

Taking the skew-symmetric part of this equation with respect to \( k \) and \( j \) and taking account of

\[ K_{ktjp} \phi_i^t \phi_h^p - K_{jtkp} \phi_i^t \phi_h^p = K_{ktjp}(\phi_i^t \phi_h^p - \phi_i^p \phi_h^t) = \Omega_{kjih}, \]

where \( \Omega_{kjih} \) are defined by (2.17), we find

\[ (3.17) \quad 2K_{kjih} + K_{kjih} + \phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} + 2 \phi_{hi} \phi_{kj} - K_{khji} + K_{jhki} \]

\[ - 3g_{kh} g_{ji} + 3g_{jh} g_{ki} + g_{ki} \eta_j \eta_h - g_{ji} \eta_k \eta_h \]

\[ = 4[\gamma_{jh} L_{ki} - \gamma_{kh} L_{ji} + \gamma_{ki} L_{jh} - \gamma_{ji} L_{kh} - \phi_{hk} M_{ij} + \phi_{kj} M_{ik} - \phi_{ji} M_{kh} + \phi_{ki} M_{jk} - \phi_{hi} M_{kj} - 2 \phi_{jk} M_{ih} + (\gamma_{ki} \eta_j - \gamma_{ji} \eta_k) \eta_h] \]

\[ + 3[(\gamma_{jh} \eta_k - \gamma_{kh} \eta_j) \eta_i + \gamma_{ki} \gamma_{jh} - \gamma_{ji} \gamma_{kh} + \phi_{ji} \phi_{hk} - \phi_{ki} \phi_{hj} + 2 \phi_{jk} \phi_{hi}]. \]

Transvecting (3.5) with \( \phi_k^j \) and taking account of (3.6), we have

\[-L_{kt} \phi_i^t = L_{ji} \phi_k^j,\]

thus we have from (3.16)

\[-M_{ki} = M_{ik}.\]

From (3.17), we have

\[ (3.18) \quad K_{kjih} = -\gamma_{kh} L_{ji} + \gamma_{jh} L_{ki} - L_{kh} \gamma_{ji} + L_{jh} \gamma_{ki} - \phi_{kh} M_{ji} + \phi_{jh} M_{ki} - M_{kh} \phi_{ji} + M_{jk} \phi_{ki} + 2(M_{kj} \phi_{ih} + \phi_{kj} M_{th}) \]

\[ - (\phi_{kh} \phi_{ji} - \phi_{jh} \phi_{ki} - 2 \phi_{kj} \phi_{ih}). \]

Transvecting (3.18) with \( g^{kh} \), we find

\[ K_{ji} + 2m L_{ji} - (\gamma_j^h L_{hi} + \gamma_i^h L_{jh}) + L_{ji} - 3(\phi_j^h M_{hi} + \phi_i^h M_{jh}) - 3 \phi_j^h \phi_{hi} = 0, \]
where 
\[ L = g^{kh}L_{kh}, \]
from which, substituting (3.16),
\[ K_{ji} + (2m + 4)g_{ji} + (L + 3)\gamma_{ji} + 4\eta_j\eta_i = 0, \]
that is,
\[ L_{ji} = -\frac{1}{2(m + 2)}[K_{ji} + (L + 3)\kappa_j - (L - 1)\eta_i], \]
and
\[ L = -\frac{K + 2(3m + 2)}{4(m + 1)}. \]
Thus, (3.18) gives
\[ B_{kji}^i = 0. \]
Thus, we have the following.

**Theorem.** In order that contact Bochner curvature tensor of a \((2m + 1)\)-dimensional Sasakian manifold \((2m + 1 > 3)\) vanishes, it is necessary and sufficient that
\[ K(X) = -\frac{1}{(g_{kj} - \eta_k\eta_j)}X_kX_j = 8(L_{ih} + \eta_i\eta_h)X^iX^h - 3, \]
where \(K(X)\) is the \(C\)-holomorphic sectional curvature with respect to a section spanned by vectors \(\phi X\) and \(\phi^2 X\) and \(L_{ih}X^iX^h\) is a certain quadratic form whose coefficients satisfy
\[ L_{ij}\phi^s_j\phi^t_i = L_{ji} + \eta_j\eta_i. \]

**References**


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