THE ASYMPTOTIC BEHAVIOR OF THE SRIVASTAVA HYPERGEOMETRIC SERIES $H_C$
NEAR THE BOUNDARY OF ITS CONVERGENCE REGION

Megumi Saigo and H. M. Srivastava

In a series of papers by Saigo et al. ([2] to [14]), there have appeared numerous properties exhibiting the behaviors of various hypergeometric functions near the boundaries of the regions of convergence of the series defining these functions. The present paper is concerned with the triple hypergeometric series

\[(1) \quad H_C(\alpha, \beta, \beta'; \delta; x, y, z) = \sum_{m,n,k=0}^{\infty} \frac{(\alpha)_{m+k}(\beta)_{m+n}(\beta')_{n+k} x^m y^n z^k}{(\delta)_{m+n+k} m! n! k!}, \quad (\max[|x|,|y|,|z|] < 1),\]

which was introduced by Srivastava (cf. [15]). The triple series (1) is expressed as a single series involving the Gauss series

\[(2) \quad H_C = \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n}(\alpha)_{m}(\beta')_{n}}{(\delta)_{m+n}} F(\alpha + m, \beta' + n; \delta + m + n; z) \frac{x^m y^n}{m! n!}.\]

We investigate the behavior of the series $H_C$ near the side $z = 1$ of the unit cube defining its convergence region. This series near the other sides $x = 1$ and $y = 1$ can be treated similarly. By virtue of (2) and the relation [1]

\[F(\alpha, \beta; \alpha + \beta; z) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(n!)^2} (1 - z)^n\]

\[(3) \quad \cdot[2\psi(n + 1) - \psi(\alpha + n) - \psi(\beta + n) - \log(1 - z)],\]

Received August 17, 1991.
we have

\( H_C(a, b, b'; a + b'; x, y, 1 - \rho) \)

\[ = \sum_{m, n=0}^{\infty} \frac{(b)_{m+n}(a)_m(b')_n}{(a + b')_{m+n}} F(a + m, b' + n; a + b' + m + n; 1 - \rho) \frac{x^m y^n}{m! n!} \]

\[ = \frac{\Gamma(a + b')}{\Gamma(a) \Gamma(b')} \sum_{m, n, k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m+k}(1)_{k+k}} [2\psi(k + 1) \]

\[-\psi(a + m + k) - \psi(b' + n + k) - \log \rho \frac{x^m y^n \rho^k}{m! n! k!} \]

\[ = \frac{\Gamma(a + b')}{\Gamma(a) \Gamma(b')} \{2S_1 - S_2 - S_3 - S_4\}, \text{ say} . \]

Let us first consider the series \( S_4 \). Dividing the series \( S_4 \) into two parts with \( k = 0 \) and \( k \geq 1 \), we obtain

\[ S_4 = \log \rho \sum_{m, n=0}^{\infty} \frac{(b)_{m+n} x^m y^n}{m! n!} \]

\[ + ab' \rho \log \rho \sum_{m, n, k=0}^{\infty} \frac{(a + 1)_{m+k}(b)_{m+n}(b' + 1)_{n+k}}{(a)_{m+k}(b')_{n+k}(2)_{k+k}} \frac{x^m y^n \rho^k}{m! n! \rho^k} \]

\[ = (1 - x - y)^{-b} \log \rho + o(1), \quad (\rho \to +0). \]

To investigate the behavior of the series \( S_1 \), we use the formula

\[ \psi(k + 1) = -\gamma + \sum_{m=1}^{k} \frac{1}{m}, \]

where \( \gamma \) is the Euler-Mascheroni constant, and we have

\[ S_1 = \sum_{m, n, k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m+k}(b')_{n+k}(1)_{k+k}} \frac{x^m y^n \rho^k}{m! n! k!} \left( -\gamma + \sum_{l=0}^{k-1} \frac{(1)_{l}}{(2)_{l}} \right) \]

\[ = -\gamma (1 - x - y)^{-b} \]

\[ + \sum_{m, n=0}^{\infty} \sum_{k=1}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m+k}(b')_{n+k}(1)_{k+k}} \sum_{l=0}^{k-1} \frac{(1)_{l}}{(2)_{l}} \frac{x^m y^n \rho^k}{m! n! k!} + o(1) \]

\[ = -\gamma (1 - x - y)^{-b} + o(1), \quad (\rho \to +0). \]

Similarly, the relation [2]

\[ \psi(z + k) = \psi(z) + \sum_{m=0}^{k-1} \frac{1}{z + m}, \quad (k = 1, 2, 3, \ldots) \]
implies that

\[ S_2 = \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m}(b')_{n}(1)^k} \cdot (\psi(a) + \frac{1}{a} \sum_{l=0}^{m+k-1} \frac{(a)_l}{(a+1)_l} \frac{x^m y^n \rho^k}{m! n! k!} \]

\[ = \psi(a) \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m}(b')_{n}(1)^k} \frac{x^m y^n \rho^k}{m! n! k!} + \frac{1}{a} \sum_{m,n,k=0}^{\infty} \sum_{l=0}^{m+k-1} \frac{(a)_l}{(a+1)_l} \frac{x^m y^n \rho^k}{m! n! k!} \]

\[ = \psi(a) \left(1 - x - y \right)^{-b} \]

\[ + \frac{b}{a} \sum_{m=0}^{\infty} \sum_{l=0}^{m} \frac{(b+1)m}{(2)_m} \frac{(a)_l}{(a+1)_l} x^m \sum_{n=0}^{\infty} \frac{(b+1+m)n}{n!} y^n + o(1) \]

\[ = \psi(a) \left(1 - x - y \right)^{-b} \]

\[ + \frac{b}{a} \sum_{p,l=0}^{\infty} \frac{(b+1)p+l}{(2)p+l(a+1)_l} x^{p+1} \left(1 - y \right)^{-b-1-p-l} + o(1) \]

\[ = \psi(a) \left(1 - x - y \right)^{-b} \]

\[ + \frac{b}{a} x \left(1 - y \right)^{-b-1} \text{F}_{1:1;1:0}^{1:2;1} \left[ \begin{array}{c} b+1 : a, 1 ; 1 ; x \\ 2 : a+1 ; -1 - y \end{array} \right]_{-b-1} + o(1), \quad (\rho \to +0), \]

where we have used the binomial expansion

\[ \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n = (1 - z)^{-\alpha} \quad (|z| < 1), \]

and the Kampé de Fériet series \( \text{F}_{1:1;1:0}^{1:2;1} \) is defined by

\[ \text{F}_{1:1;1:0}^{1:2;1} \left[ \begin{array}{c} \alpha : \beta, \beta' ; \eta ; x, y \\ \delta : \epsilon ; -i \end{array} \right] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_m(\eta)_n}{(\delta)_{m+n}(\epsilon)_m} \frac{x^m y^n}{m! n!}, \]

which converges when \( \max\{|x|, |y|\} < 1 \) (cf. [15] for a detailed description of the Kampé de Fériet series).

Similar arguments show that

\[ S_3 = \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_{m}(b')_{n}(1)^k} \left( (\psi(b') + \frac{1}{b'} \sum_{l=0}^{n+k-1} \frac{(b')_l}{(b'+1)_l} \frac{x^m y^n \rho^k}{m! n! k!} \right) + o(1), \]

where we have used the binomial expansion
\( \psi(b')(1 - x - y)^{-b} \)
\( + \frac{b}{b'} y \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{(b + 1)_n(b')_i}{(b' + 1)_i(2)_n} y^n \sum_{m=0}^{\infty} \frac{(b + 1 + n)_m}{m!} x^m + o(1) \)
\( = \psi(b')(1 - x - y)^{-b} \)
\( + \frac{b}{b'} y \sum_{r,l=0}^{\infty} \frac{(b + 1)_{r+l}(b')_l}{(b' + 1)_l(2)_{r+l}} y^{r+l}(1 - x)^{-b-1-r-l} + o(1) \)
\( = \psi(b')(1 - x - y)^{-b} \)
\( + \frac{b}{b'} y(1 - x)^{-b-1} F_{1;1:1}^{1:2;1} \left[ \begin{array}{c} b + 1 : b', 1; 1; y \\ \frac{1}{2} : b' + 1; -; 1 - x, 1 - x \end{array} \right] + o(1), \quad (\rho \to +0). \)

Thus the relations (4) to (9) yield the following formula exhibiting the behavior of the Srivastava series near the side \( z = 1 \) of the unit cube defining its convergence region

\( H_C(a, b, b'; a + b'; x, y, 1 - \rho) \)
\( = -\frac{\Gamma(a + b')}{\Gamma(a)\Gamma(b')} \left\{ (1 - x - y)^{-b}[2\gamma + \psi(a) + \psi(b') + \log \rho] \right\} \)
\( + \frac{b}{a} x(1 - y)^{-b-1} F_{1;1:1}^{1:2;1} \left[ \begin{array}{c} b + 1 : a, 1; 1; x \\ \frac{1}{2} : a + 1; -; 1 - y, 1 - y \end{array} \right] \)
\( + \frac{b}{b'} y(1 - x)^{-b-1} F_{1;1:1}^{1:2;1} \left[ \begin{array}{c} b + 1 : b', 1; 1; y \\ \frac{1}{2} : b' + 1; -; 1 - x, 1 - x \end{array} \right] \}
+ o(1), \quad (\rho \to +0). \)

Acknowledgement. The present investigation was initiated during the second author's visit to Fukuoka University in October 1990 while he was on subbatical leave from the University of Victoria. This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

Behavior of Srivastava Series


Department of Applied Mathematics, Fukuoka University, Fukuoka 814-01, Japan.

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada.